Spherical model of growing interfaces

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Abstract

Building on an analogy between the ageing behaviour of magnetic systems and growing interfaces, the Arcetri model, a new exactly solvable model for growing interfaces is introduced, which shares many properties with the kinetic spherical model. The long-time behaviour of the interface width and of the two-time correlators and responses is analysed. For all dimensions $d \neq 2$, universal characteristics distinguish the Arcetri model from the Edwards-Wilkinson model, although for d>2 all stationary and non-equilibrium exponents are the same. For d=1 dimensions, the Arcetri model is equivalent to the p=2 spherical spin glass. For 2 < d < 4 dimensions, its relaxation properties are related to the ones of a particle-reaction model, namely a bosonic variant of the diffusive pair-contact process. The global persistence exponent is also derived.

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1 Introduction

The physics of the growth of interfaces is a paradigmatic example of the emergence of non-equilibrium cooperative phenomena [1, 2, 3, 4, 5, 6, 7]. The morphological evolution of the growing interface is characterised by its fractal properties as well as several growth and roughness exponents. More recently, aspects of the time-dependent evolution of the interface have been investigated and have been found to be quite analogous to phenomena encountered in the physical ageing in glassy and non-glassy systems [8, 9]. It has been understood that growing interfaces can be cast into distinct universality classes, including those associated to the Edwards-Wilkinson (EW) equation [10], where the relaxation is dominated by the linear interface tension, and the Kardar-Parisi-Zhang (KPZ) equation [11], which includes as well a non-linear effect of the local slope.

The exponents of growing interfaces are defined as follows. The basic quantity is the time-space-dependent local height $h(t, \mathbf{r})$ and one is mainly interested in the fluctuations around the spatially averaged height $\overline{h}(t) := L^{-d} \sum_{\mathbf{r} \in \Lambda} h(t, \mathbf{r})$; for notational convenience defined here on a hyper-cubic lattice $\Lambda \subset \mathbb{Z}^d$ with L^d sites and the sum runs over the lattice sites. A central quantity is the interface width

$$w^{2}(t;L) := \frac{1}{L^{d}} \sum_{\boldsymbol{r} \in \Lambda} \left\langle \left(h(t,\boldsymbol{r}) - \overline{h}(t) \right) \right\rangle^{2} = L^{2\alpha} f_{w} \left(tL^{-z} \right) \sim \begin{cases} t^{2\beta} & ; & \text{if } tL^{-z} \ll 1 \\ L^{2\alpha} & ; & \text{if } tL^{-z} \gg 1 \end{cases}$$
(1.1)

and we recall the generically expected Family-Viscek scaling form [12], where α is the roughness exponent, β the growth exponent and $z = \alpha/\beta$ the dynamical exponent. Herein, $\langle . \rangle$ denotes an average over many independent samples (under the same thermodynamic conditions). Throughout this work, we shall consider the $L \to \infty$ limit. The initial state will always be a flat, uncorrelated substrate. Relaxational properties of the interface can be characterised by the two-time correlations and (linear) responses

$$C(t,s;\mathbf{r}) := \left\langle \left(h(t,\mathbf{r}) - \left\langle \overline{h}(t) \right\rangle \right) \left(h(s,\mathbf{0}) - \left\langle \overline{h}(s) \right\rangle \right) \right\rangle = s^{-b} F_C \left(\frac{t}{s}; \frac{\mathbf{r}}{s^{1/z}} \right)$$
(1.2)

$$R(t,s;\boldsymbol{r}) := \frac{\delta \left\langle h(t,\boldsymbol{r}) - \overline{h}(t) \right\rangle}{\delta j(s,\boldsymbol{0})} \bigg|_{j=0} = \left\langle h(t,\boldsymbol{r})\widetilde{h}(s,\boldsymbol{0}) \right\rangle = s^{-1-a} F_R\left(\frac{t}{s};\frac{\boldsymbol{r}}{s^{1/z}}\right) \quad (1.3)$$

where spatial translation-invariance has been implicitly admitted and j is an external field conjugate to h. Furthermore, in the context of Janssen-de Dominicis theory, h is the conjugate response field to h [13, 7]. In the long-time limit, where both $t, s \gg \tau_{\text{micro}}$ and $t - s \gg \tau_{\text{micro}}$ (τ_{micro} is a microscopic reference time), generalised Family-Viscek scaling forms were assumed [14, 15, 16, 17, 18] which are analogous to the scaling forms of 'simple ageing' in other non-equilibrium systems [8, 19, 9]. The exponent $b = -2\beta$ [14, 20], but the relationship of a to other exponents seems to depend on the universality class [18]. Finally, the autocorrelation exponent λ_C and the autoresponse exponent λ_R are defined from the asymptotics $F_{C,R}(y, \mathbf{0}) \sim y^{-\lambda_{C,R}/z}$ as $y \to \infty$. There is a bound $\lambda_C \geq (d+zb)/2$, see appendix B. Field-theoretical considerations in ageing simple magnets with a non-conserved model-A dynamics and with disordered initial conditions strongly indicate that the non-equilibrium exponents λ_C , λ_R should be independent

¹Applications include settings as distinct as deposition of atoms on a surface, solidification, flame propagation, population dynamics, crack propagation, chemical reaction fronts or the growth of cell colonies.

of those describing the stationary state [13, 21, 7]. This is different, however, for the KPZ universality class, where for dimensions d < 2 it was shown that $\lambda_C = d$, to all orders in perturbation-theory [22]. However, for $d \geq 2$ there exists a strong-coupling fixed point of the KPZ equation which cannot be reached by a perturbative analysis, see e.g. [23, 24, 7]; but which can be studied through non-perturbative renormalisation-group techniques [25, 26].

Known estimates of all these exponents, for several universality classes, are listed in table 1. In most of the quoted experiments, two-time autocorrelators such as $C(t,0;\mathbf{0})$ were also used to extract β , these approaches rely on the relation $b = -2\beta$. The study of two-time global quantities, also of interest for experiments, gives a different access to the exponents, especially β [41, 42]. For the experiments reported in [36, 33, 32], merely an average of the values quoted in the sources is included in table 1. While the theoretical analysis, at least for the exponents α, β, z in the EW and KPZ classes, has been achieved long ago [10, 11], recent advances in experimental techniques have permitted to obtain precise and reliable estimates of these exponents, independently and in several distinct systems,³ and in good agreement with each other and with the theoretical predictions.⁴ Notably, the great variety of systems for which the exponents z, β, α are consistent with the KPZ-equation, clearly attest its universality, at least for 1D systems. For the 1D (positive) quenched KPZ class, a mapping onto 1D directed percolation was proposed [38]. This produces $\alpha = \nu_{\perp}/\nu_{\parallel} \simeq 0.63261$ which apparently fits the experimental and simulational data well (using the very precisely known values of the exponents $\nu_{\parallel,\perp}$ of directed percolation, see e.g. [45]). However, other predictions following from this mapping, such as z=1 [38, 1], do not seem to be generically reproduced by the presently available evidence. In several of these experiments, universal amplitude ratios have been measured as well [37, 46, 32, 35]. For further details on procedures and the extraction of the exponents, we refer to the quoted sources. For a recent thorough review on experimental results, see [47].

In d=1 dimension, several further notable advances have been achieved. These concern a remarkable exact solution of the KPZ equation and the spectacular relationship of the probability distribution $\mathcal{P}(h)$ of the fluctuation $h-\overline{h}$ with the extremal value statistics of the largest eigenvalue of random matrices, see [48, 49, 50, 51, 52, 24], and its successful experimental confirmation [46, 32, 47, 35, 28, 53]. Still, this progress seems to rely on specific properties of the one-dimensional case. One may ask, if there might exist alternative routes to a further conceptual understanding, less dependent on the particular mathematical circumstances which render the 1D KPZ equation analytically treatable. Here, an analogy with magnetic systems at their critical point might be helpful. Some elements of such an analogy are sketched in table 2. Indeed, several observables derived from the main quantity, namely the order parameter $\phi(t, \mathbf{r})$ or the interface height $h(t, \mathbf{r})$, respectively, show an analogous dynamical behaviour. In particular, the exponent b, to be read off from the scaling behaviour (1.2) of the autocorrelator, is $b = 2\beta/(\nu z)$ for a critical magnet and $b = -2\beta$ for interfaces. This is consistent with the scaling behaviour of the equal-time variance/interface width, respectively. On the other hand, for magnets relaxing towards an equilibrium state, a fluctuation-dissipation theorem gives $a=b=2\beta/(\nu z)$, see e.g. [9], whereas no such generic result is known for the fundamentally non-equilibrium interfaces. The analogy between magnetic systems and interfaces is further

²However, this relation needs no longer to hold true when $d > d^*$.

³In [33], colonies of both benign and malign cells, for both flat and circular interfaces were measured.

⁴See [43, 44, 29] for recent estimates, of the exponents α , β and also of universal amplitude ratios, in the KPZ universality class, up to d = 11.

⁵In the EW model, one has a = b [15], while in the 1D KPZ model 1 + a = b + 2/z [9] and the relationship

Table 1: Exponents of growing interfaces in four universality classes, namely Kardar-Parisi-Zhang (KPZ), (positive) quenched KPZ (QKPZ), Edwards-Wilkinson (EW) and Arcetri (for both $T = T_c$ and $T < T_c$) are represented. In each group, labelled by the universality class, first are indicated available exact results[†] and/or recent simulational estimates.[†] Then follow experimental results, where the system's dimension d and the nature of the interface are indicated. The numbers in bracket give the error in the last digit(s).

*The scaling relation $\alpha = z\beta$, resp. $\alpha + z = 2$ for the KPZ class, was used to complete the entries as much as possible, with an error estimated from the relative errors of the exponent(s) given in the source.

model	d	z	β	α	a	b	λ_C	λ_R	Ref.
KPZ	1	3/2	1/3	1/2	-1/3	-2/3	1	1	[11, 22, 18]
	2	$1.61(2)^*$	0.2415(15)	0.393(4)	0.30(1)	-0.483(3)	1.97(3)	2.04(3)	[27]
	2	$1.61(2)^*$	0.241(1)	0.393(3)		-0.483	1.91(6)		[28]
	2	1.61(5)	0.244(2)	0.369(8)					[29]
	2	$1.627(4)^*$	$0.229(6)^*$	0.373(3)					[26]
Ag electrodeposition [†]	1		$\approx 1/3$	$\approx 1/2$					[30]
slow paper combustion [†]	1	$1.44(12)^*$	0.32(4)	0.49(4)					[31]
liquid crystal [†]	1	$1.34(14)^*$	0.32(2)	0.43(6)		$\approx -2/3$	≈ 1		[32]
liquid crystal [‡]	1	$1.44(10)^*$	0.334(3)	0.48(5)		$\approx -2/3$	0?		[32]
cell colony growth ^{†‡}	1	1.56(10)	0.32(4)	0.50(5)					[33, 34]
(almost) isotrope colloïds [‡]	1		0.37(4)	0.51(5)					[35]
autocatalytic reaction front [†]	1	$1.45(11)^*$	0.34(4)	0.50(4)					[36]
$\mathrm{CdTe/Si}(100) \; \mathrm{film}^{\dagger}$	2	1.61(5)	0.24(4)	$0.39(5)^*$					[37]
QKPZ	1		≈ 0.9	0.63261					[38, 39]
cell colony growth (disordered) [†]	1	0.84(5)	0.75(5)	0.63(4)					[34]
autocatalytic reaction front [†]	1		0.61(5)	0.66(4)					[36]
strongly ellipsoid colloïds [‡]	1		0.68(5)	0.61(2)					[35]
EW	< 2	2	(2-d)/4	(2-d)/2	d/2 - 1	d/2 - 1	d	d	
	2	2	$0(\log)^{\#}$	$0(\log)^{\#}$	0	0	2	2	[10, 15]
	> 2	2	0	0	d/2 - 1	d/2 - 1	d	d	
sedimentation/electrodispersion	2		$0(\log)^{\#}$	$0(\log)^{\#}$					[40]
Arcetri $T = T_c$	< 2	2	(2-d)/4	(2-d)/2		d/2 - 1	3d/2 - 1	3d/2 - 1	
	2	2	$0(\log)^{\#}$		0	0	2	2	
	> 2	2	0	0	d/2 - 1	d/2 - 1	d	d	
$T < T_c$	d	2	1/2	1	d/2 - 1	-1	d/2 - 1	d/2 - 1	

[†] flat interface. ‡ circular interface. #For d=2, one has $w(t;L) \sim \sqrt{\ln t} f_w(\ln L/\ln t)$.

Table 2: Analogies between the critical dynamics in magnets and growing interfaces. Several observables derived from the central physical quantity, the order parameter/height, respectively, are shown, where h or j, respectively, are conjugate to the order parameter/height. The average $\langle . \rangle_c$ denotes a connected correlator. Some models, with the equilibrium hamiltonian for magnets, are defined through their kinetic equations, where η is a standard white noise, Δ the spatial laplacian and D, g, ν, μ are constants. The existence of known exact solutions in the Ising and KPZ models is indicated.

	magnets	interfaces					
order parameter	$\phi(t, \boldsymbol{r})$	height	$h(t, oldsymbol{r})$				
variance	$\langle (\phi(t, \boldsymbol{r}) - \langle \phi(t, \boldsymbol{r}) \rangle)^2 \rangle \sim t^{-2\beta/(\nu z)}$	width	$w^2(t) = \langle (h(t, \mathbf{r}) - \overline{h}(t))^2 \rangle \sim t^{2\beta}$				
autocorrelator	$C(t,s) = \langle \phi(t, \mathbf{r})\phi(s, \mathbf{r}) \rangle_c$	autocorrelator	$C(t,s) = \langle h(t, \boldsymbol{r}) h(s, \boldsymbol{r}) \rangle_c$				
autoresponse	$R(t,s) = \delta \langle \phi(t, \boldsymbol{r}) \rangle / \delta h(s, \boldsymbol{r}) _{h=0}$	autoresponse	$R(t,s) = \delta \langle h(t, \mathbf{r}) \rangle / \delta j(s, \mathbf{r}) _{j=0}$				
Models:			-				
gaussian field	$\mathcal{H}[\phi] = -\frac{1}{2} \int \!\mathrm{d} m{r} (abla \phi)^2$	EW					
	$\partial_t \phi = D\Delta \phi + \eta$		$\partial_t h = \nu \Delta h + \eta$				
Ising model	$\mathcal{H}[\phi] = -\frac{1}{2} \int d\mathbf{r} \left[(\nabla \phi)^2 + \frac{g}{2} \phi^4 \right]$	KPZ					
	$\partial_t \phi = D(\tilde{\Delta}\phi + g\phi^3) + \eta$		$\partial_t h = \nu \Delta h + \frac{\mu}{2} (\nabla h)^2 + \eta$				
Exact solutions for the Ising and KPZ universality classes:							
equilibrium $d =$	2 [54, 55, 56]						
relaxation $d =$	1 [57]	relaxation $d =$	[48, 49, 51]				

illustrated by the form of the kinetic equations of the time-dependent order parameter, schematically written as $\partial_t \phi = -D\delta \mathcal{H}/\delta\phi + \eta$, where η is a gaussian white noise. Clearly, the gaussian field/EW model share mean-field characteristics, whereas the behaviour of the Ising and KPZ models is determined by the non-linearities in their equations of motion. These non-linearities make exact solutions so difficult to find: for instance, in the 2D Ising model at equilibrium, only the cases (i) of a vanishing magnetic field h=0 and arbitrary temperature T [54] and (ii) of the fixed temperature $T=T_c$ and an arbitrary magnetic field h [55, 56] have been solved (see [58] for a recent overview of numerical and experimental tests).

In magnetic phase transitions, in view of these technical difficulties, the so-called spherical model was originally proposed by Berlin and Kac [59] in order to be able to explore generic properties of equilibrium phase transitions in the context of a non-trivial exactly solvable model. A simple way to introduce it is to replace the discrete Ising spins $\sigma_i = \pm 1$, attached to each site i of the lattice Λ , by continuous spin variables $\sigma_i \mapsto s_i \in \mathbb{R}$ subject to the 'spherical constraint' $\sum_{i \in \Lambda} s_i^2 \stackrel{!}{=} \mathcal{N}$, where \mathcal{N} is the number of sites of the lattice. Especially in the form of the 'mean spherical model' (where the spherical constraint is only required on average [60], which considerably shortens the calculations, without modifying the critical behaviour), the spherical model has become an often-used test case in many distinct situations. For the most common case of short-ranged interactions, a phase transition with a critical temperature $T_c > 0$ exists in dimensions d > 2. For dimensions 2 < d < 4, the equilibrium exponents are different from the mean-field theory of the free gaussian field [59]. For the dynamics, and in the continuum limit, this amounts in the kinetic equation of motion to replace the non-linearity by a more simple

term, viz. $\phi^3 \mapsto \langle \phi^2 \rangle \phi =: \mathfrak{z}(t)\phi$ and to fix the Lagrange multiplier $\mathfrak{z}(t)$ through the spherical constraint. This procedure can be applied to study the relaxational dynamics and to extract the exponents of both equilibrium and non-equilibrium critical dynamics, and was performed many times, see e.g. [61, 62, 63, 64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79, 80, 81, 82] and references therein. For reviews, see [83, 84, 85, 19, 9, 7].

Here, we shall inquire whether a spherical model variant can be sensibly defined for models describing growing interfaces. In particular, we shall discuss the following questions:

- 1. which property of the interface heights could be construed to take only values ± 1 in order to identify a sensible 'spherical approximation'?
- 2. does one obtain a non-trivial model, distinct from the mean-field-like EW class? And does there exist⁶ a well-defined upper critical dimension d^* ?
- 3. in spite of the analogy with magnetic systems relaxing towards equilibrium, is the relaxation process an equilibrium or a non-equilibrium one?

This work is organised as follows: section 2 defines the Arcetri model and section 3 outlines the exact calculation of the interface width and of the two-time responses and correlators. The 1D model is shown to be equivalent to the p=2 spherical spin glass. In section 4, the long-time behaviour is explicitly found. We discuss in detail the non-equivalence with the EW-model for all dimensions $d \neq 2$; the relationship of the relaxation behaviour with the one of a bosonic particle-reaction model for dimensions 2 < d < 4; and comment on the global persistence probability. We conclude in section 5. Details of the calculations are treated in appendix A. The Yeung-Rao-Desai inequality is revisited in appendix B.

2 The Arcetri model

We begin by stating the definition of the model⁷ to be analysed, first in d = 1 dimensions. Consider a set of height variables $h_n(t)$ attached to the sites n of a ring with N sites. In what follows, we shall work with its (discrete, symmetrised) derivatives

$$u_n(t) := \frac{1}{2} \left(h_{n+1}(t) - h_{n-1}(t) \right) \tag{2.1}$$

Furthermore, to each lattice site one attaches a gaussian random variable $\eta_n(t)$, with the moments

$$\langle \eta_n(t) \rangle = 0 , \langle \eta_n(t) \eta_m(t') \rangle = 2\Gamma T \delta(t - t') \delta_{n,m}$$
 (2.2)

The defining equations of motion for the 1D Arcetri model are

$$\partial_t u_n(t) = \Gamma \left(u_{n+1}(t) + u_{n-1}(t) - 2u_n(t) \right) + \mathfrak{z}(t)u_n(t) + \frac{1}{2} \left(\eta_{n+1}(t) - \eta_{n-1}(t) \right) \tag{2.3}$$

$$\sum_{n=0}^{N-1} \left\langle u_n(t)^2 \right\rangle = N \tag{2.4}$$

⁶The existence of a finite d^* would be analogous to critical magnets, whereas for the KPZ equation the question is still unresolved, although numerical results suggest that d^* might be infinite, see [43, 44, 29].

⁷The name is inspired by the *lieu* where this model was conceived and this work was done.

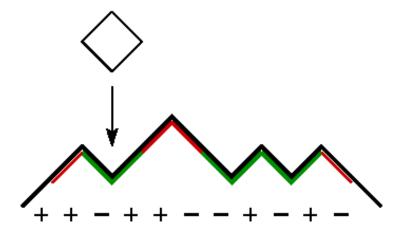


Figure 1: Schematic illustration of a 1D growing interface respecting the RSOS constraint that the local slopes $u_{i+1/2} = h_{i+1} - h_i \stackrel{!}{=} \pm 1$ on nearest-neighbour sites. The interface grows through sequential absorption of square-shaped particles. Those 'active' positions into which an adsorption is possible are underlined in green and those 'inactive' ones where this is prohibited by the RSOS constraint are underlined in red. Active and inactive positions have to be updated after each adsorption process.

where the Lagrange multiplier⁸ $\mathfrak{z}(t)$ is determined from the 'spherical constraint' (2.4) and Γ and T are constants.

Our motivation to study this model comes from the well-known representation of lattice realisations of the KPZ universality class in terms of the totally asymmetric exclusion model (TASEP) [11, 20, 7]. In figure 1 an interface is sketched, and its growth occurs through the adsorption of additional blocks, one at a given time, but such that between neighbouring sites the RSOS-condition $h_{i+1}(t) - h_i(t) = \pm 1$ is obeyed. Below the height configuration, the value of the interface $u_{i+1/2} = h_{i+1} - h_i$ on the dual lattice (the links) is indicated. Let $u_{i+1/2} = -1 \mapsto \bullet$ represent a dual site occupied by a particle and $u_{i+1/2} = +1 \mapsto \circ$ an empty dual site. Then the adsorption process corresponds to an irreversible change $\bullet \circ \to \circ \bullet$ of a particle hopping to the right between two neighbouring sites of the dual lattice, whereas the inverse reaction $\circ \bullet \to \bullet \circ$ is not admitted. In the continuum limit, this process is described by the KPZ equation, as given in table 2.9

In our search for a spherical model analogue for the KPZ equation, we shall therefore identify the slopes (2.1) as the variables on which one could impose a spherical condition (2.4). Then, it should be preferable to consider the equation of motion of the slope, which in the continuum limit becomes $u(t, x) = \partial_x h(t, x)$ and obeys the noisy Burgers equation

$$\partial_t u(t,x) = \nu \partial_x^2 u(t,x) + \mu u(t,x) \partial_x u(t,x) + \partial_x \eta(t,x)$$
(2.5)

Finally, we replace the non-linearity $u\partial_x u \mapsto \mathfrak{z}(t)u$ in (2.5) by a spherical model construction, analogously to the definition of the spherical model of a ferromagnet [59, 60] and fix the Lagrange multiplier $\mathfrak{z}(t)$ from the spherical constraint (2.4).¹⁰ Eqs. (2.3),(2.4) give the discretised

⁸We do not consider any fluctuations in $\mathfrak{z}(t)$, since they do not contribute to the spatially local averages [72] on which we concentrate here.

⁹See [86] for a rigorous proof in 1D. Another often-used lattice representation is the Kim-Kosterlitz model [87], where $h_{i+1} - h_i = \pm 1, 0$, such that the continuum limit is again the KPZ equation.

¹⁰In the kinetic spherical model of a ferromagnet, using a mean spherical constraint simplifies considerably

version of this construction.

Clearly, eqs. (2.4),(2.5) are identical to those of the conventional spherical model, see e.g. [65, 69], but with the standard white noise $\eta(t,x)$ replaced by its derivative $\partial_x \eta(t,x)$. This new feature brings down the upper critical dimension from $d^* = 4$ to $d^* = 2$ [1]. It is more surprising to find that the mere modification of the noise is enough to render our model identical to the spherical spin glass [64], as we shall show in section 2.4. On the other hand, the present work with its specific way of introducing the spherical constraint is merely meant as a first step in a more systematic exploration of 'spherical variants' of equations such as (2.5). We shall come back in section 5 to other possibilities which might keep a more clear memory of the non-linear terms in (2.5). Higher-order non-linearities will either turn out to be irrelevant or else may create some 'multicritical' variant of scale-invariant growth [1, 2, 3].

In order to see how to generalise this model to any dimension $d \neq 1$, we briefly consider the 2D case, for simplicity of notation. The height on each site is $h_{n,m}(t)$. Then we have slopes in each of the two lattice directions, namely

$$u_{n,m}(t) := \frac{1}{2} \left(h_{n+1,m}(t) - h_{n-1,m}(t) \right) , \quad v_{n,m}(t) := \frac{1}{2} \left(h_{n,m+1}(t) - h_{n,m-1}(t) \right)$$
 (2.6)

which obey the equations of motion (for brevity, the dependence on t is suppressed)

$$\partial_{t}u_{n,m} = \Gamma\left(u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1} - 4u_{n,m}\right) + \mathfrak{z}(t)u_{n,m} + \frac{1}{2}\left(\eta_{n+1,m} - \eta_{n-1,m}\right)$$

$$\partial_{t}v_{n,m} = \Gamma\left(v_{n+1,m} + v_{n-1,m} + v_{n,m+1} + v_{n,m-1} - 4v_{n,m}\right) + \mathfrak{z}(t)v_{n,m} + \frac{1}{2}\left(\eta_{n,m+1} - \eta_{n,m-1}\right)$$
(2.7)

Considering the spherical constraint, we must count how many links exist on a torus $\mathbb{S}^1 \otimes \mathbb{S}^1$ with $N \times N = N^2$ sites. Since on each line of the lattice, one has N links, there are N such lines and there two dimensions along with the links must be counted, there is a total of $N \times N \times 2 = 2N^2$ links. Hence, the spherical constraint must be

$$\sum_{n,m=0}^{N-1} \left\langle \left(u_{n,m}^2 + v_{n,m}^2 \right) \right\rangle = 2N^2 \tag{2.8}$$

The extension to generic d is now obvious and will be written down explicitly in the next section.¹¹

Clearly, in analogy with the EW and KPZ equations, we have already performed implicitly a Galilei transformation and shall always work in a reference frame which moves upwards with the average rate of particle deposition. The parameter T in the noise correlator serves to distinguish the kinetic noise from the effective diffusion constant ν .

the calculations, [65]. Provided the infinite-system limit $N \to \infty$ is taken before the large-time limit, the consideration of a non-averaged spherical constraint leads to the same results for the long-time behaviour [68].

¹¹The formulation chosen here is explicitly invariant under spatial rotations. Any attempt of a generalisation towards more than one spherical constraint will likely lead to a breaking of that symmetry.

3 Solution

3.1 Equations of motion

The solution of the Arcetri model equations of motion, i.e. eqs. (2.7),(2.8) for d=2, is carried out in Fourier space. In d spatial dimensions, there are d slopes, denoted $u_{a,n}(t)$, with $a=1,\ldots,d$ and $\mathbf{n}=(n_1,\ldots,n_d)$. On a torus $\mathbb{T}^d=\mathbb{S}^1\otimes\cdots\otimes\mathbb{S}^1=(\mathbb{S}^1)^{\otimes d}$, represented by a hyper-cubic lattice with N^d sites, one has the Fourier transforms

$$\widehat{u}_{a}(t, \mathbf{p}) = \sum_{n_{1}=0}^{N-1} \cdots \sum_{n_{d}=0}^{N-1} \exp\left(-\frac{2\pi i}{N} \mathbf{p} \cdot \mathbf{n}\right) u_{a, \mathbf{n}}(t)$$

$$u_{a, \mathbf{n}}(t) = \frac{1}{N^{d}} \sum_{n_{1}=0}^{N-1} \cdots \sum_{n_{d}=0}^{N-1} \exp\left(\frac{2\pi i}{N} \mathbf{p} \cdot \mathbf{n}\right) \widehat{u}_{a}(t, \mathbf{p})$$
(3.1)

Our equations of motion take the form

$$\partial_t \widehat{u}_a(t, \mathbf{p}) = -2\Gamma \omega(\mathbf{p}) \widehat{u}_a(t, \mathbf{p}) + \mathfrak{z}(t) \widehat{u}_a(t, \mathbf{p}) + i \sin\left(\frac{2\pi}{N} p_a\right) \widehat{\eta}(t, \mathbf{p})$$
(3.2)

with the dispersion relation $\omega(\mathbf{p}) = \sum_{j=1}^{d} \left[1 - \cos\left(\frac{2\pi}{N}p_j\right)\right]$, and the gaussian noise correlators

$$\langle \widehat{\eta}(t, \mathbf{p}) \rangle = 0 , \langle \widehat{\eta}(t, \mathbf{p}) \widehat{\eta}(t', \mathbf{q}) \rangle = 2\Gamma T N^d \delta(t - t') \delta_{\mathbf{p} + \mathbf{q}, \mathbf{0}}$$
 (3.3)

Since the definition (2.6) implies integrability conditions, ¹² the interface height is given by

$$\widehat{u}_a(t, \mathbf{p}) = i \sin\left(\frac{2\pi}{N}p_a\right)\widehat{h}(t, \mathbf{p}) \; ; \quad \text{if } \mathbf{p} \neq \mathbf{0}, \; a = 1, \dots, d$$
 (3.4)

We restrict attention to interfaces which on average are initially flat and uncorrelated. Indeed, the initial modes $\hat{h}(0, \mathbf{p})$ are assumed gaussian random variables with the moments

$$\langle \widehat{h}(0, \boldsymbol{p}) \rangle = N^d H_0 \, \delta_{\boldsymbol{p}, \boldsymbol{0}} , \, \langle \widehat{h}(0, \boldsymbol{p}) \widehat{h}(0, \boldsymbol{q}) \rangle = N^d H_1 \, \delta_{\boldsymbol{p} + \boldsymbol{q}, \boldsymbol{0}}$$
 (3.5)

where $H_{0,1}$ are initial parameters. The first condition (3.5) means that the average initial slopes vanish. From the second, if $H_1 = H_0^2$, the initial interface width vanishes. We shall restrict to this case below, see eq. (3.22).

The equations of motion (3.2) have the following solution, with $a = 1, \dots, d$

$$\widehat{u}_{a}(t, \mathbf{p}) = \widehat{u}_{a}(0, \mathbf{p}) \exp \left[-2\Gamma\omega(\mathbf{p})t + \int_{0}^{t} dt' \,\mathfrak{z}(t')\right] + \int_{0}^{t} d\tau \, \mathrm{i} \sin\left(\frac{2\pi}{N}p_{a}\right) \widehat{\eta}(\tau, \mathbf{p}) \exp \left[-2\Gamma\omega(\mathbf{p})(t-\tau) + \int_{\tau}^{t} dt' \,\mathfrak{z}(t')\right]$$
(3.6)

¹²From (2.6), one has in 2D that $\frac{1}{2}(u_{n,m+1}(t)-u_{n,m-1}(t))=\frac{1}{2}(v_{n+1,m}(t)-v_{n-1,m}(t))$, which in the continuum limit amounts to $\partial u/\partial y=\partial v/\partial x$. Standard calculus [88, p. 104] then shows that there exists a function h such that $u=\partial h/\partial x$ and $v=\partial h/\partial y$, on a simply connected domain. Eq. (3.4) expresses this in discrete Fourier space.

In order to work out the explicit form of the spherical constraint, we follow the lines of the magnetic spherical model and define [65]

$$g(t) := \exp\left(-2\int_0^t \mathrm{d}t' \,\mathfrak{z}(t)\right) \tag{3.7}$$

Then the spherical constraint becomes, using (3.1) and the initial condition (3.4), (3.5),

$$dN^{d} \stackrel{!}{=} \sum_{a=1}^{d} \sum_{n_{1}=0}^{N-1} \cdots \sum_{n_{d}=0}^{N-1} \left\langle u_{a,\boldsymbol{n}}(t)^{2} \right\rangle$$

$$= \sum_{p_{1}=0}^{N-1} \cdots \sum_{p_{d}=0}^{N-1} \left[H_{1} \lambda(\boldsymbol{p}) e^{-4\Gamma\omega(\boldsymbol{p})t} \frac{1}{g(t)} + \int_{0}^{t} d\tau \ 2\Gamma T \ \lambda(\boldsymbol{p}) e^{-4\Gamma\omega(\boldsymbol{p})(t-\tau)} \frac{g(\tau)}{g(t)} \right]$$
(3.8)

with $\lambda(\boldsymbol{p}) := \sum_{a=1}^{d} \sin^2\left(\frac{2\pi}{N}p_a\right)$. If we now define

$$f(t) := \frac{1}{N^d} \sum_{p_1=0}^{N-1} \cdots \sum_{p_d=0}^{N-1} \lambda(\mathbf{p}) e^{-4\Gamma t \omega(\mathbf{p})}$$
(3.9)

the spherical constraint can be written as a Volterra integral equation

$$H_1 f(t) + 2\Gamma T \int_0^t d\tau \ g(\tau) f(t - \tau) = d g(t)$$
 (3.10)

Its solution will be found in complete analogy with well-established techniques [61, 64, 65].

At this point, the infinite-size limit $N \to \infty$ can be taken. The solution (3.6) becomes

$$\widehat{u}_a(t, \mathbf{p}) = \widehat{u}_a(0, \mathbf{p})e^{-2\Gamma t\omega(\mathbf{p})} \frac{1}{\sqrt{g(t)}} + \int_0^t d\tau \, i \sin p_a \, \widehat{\eta}(\tau, \mathbf{p})e^{-2\Gamma(t-\tau)\omega(\mathbf{p})} \sqrt{\frac{g(\tau)}{g(t)}}$$
(3.11)

where the dispersion relation now reads $\omega(\mathbf{p}) = \sum_{a=1}^{d} [1 - \cos p_a]$ and $\mathbf{p} \in \mathcal{B} := [-\pi, \pi]^d$ is in the Brillouin zone. Letting now $\lambda(\mathbf{p}) := \sum_{a=1}^{d} \sin^2 p_a$, we have from (3.9),

$$f(t) = \frac{1}{(2\pi)^d} \int_{\mathcal{B}} d\mathbf{p} \, \lambda(\mathbf{p}) \, e^{-4\Gamma t \omega(\mathbf{p})}$$

$$= \frac{d}{2\pi} \int_{-\pi}^{\pi} d\mathbf{p} \, \sin^2 p \, e^{4\Gamma t \cos p} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} d\mathbf{q} \, e^{4\Gamma t \cos q} \right)^{d-1} e^{-4d\Gamma t}$$

$$= d \frac{e^{-4\Gamma t} I_1(4\Gamma t)}{4\Gamma t} \left(e^{-4\Gamma t} I_0(4\Gamma t) \right)^{d-1}$$
(3.12)

where the I_n denote modified Bessel functions [89]. The function g(t) is found from the Volterra integral equation (3.10). Furthermore, since (3.4) is now written as $\widehat{u}_a(t, \mathbf{p}) = \mathrm{i} \sin p_a \, \widehat{h}(t, \mathbf{p})$, if $\mathbf{p} \neq \mathbf{0}$, the interface height becomes¹³

$$\widehat{h}(t, \boldsymbol{p}) = \widehat{h}(0, \boldsymbol{p}) e^{-2\Gamma t \omega(\boldsymbol{p})} \frac{1}{\sqrt{g(t)}} + \int_0^t d\tau \, \widehat{\eta}(\tau, \boldsymbol{p}) e^{-2\Gamma(t-\tau)\omega(\boldsymbol{p})} \sqrt{\frac{g(\tau)}{g(t)}}$$
(3.13)

¹³The EW model is obtained by setting $g(t) = 1 \leftrightarrow \mathfrak{z}(t) = 0$ [15].

and the initial conditions (3.5) now read

$$\langle \hat{h}(0, \mathbf{p}) \rangle = H_0 \, \delta(\mathbf{p}) \, , \, \langle \hat{h}(0, \mathbf{p}) \hat{h}(0, \mathbf{q}) \rangle = H_1 \, \delta(\mathbf{p} + \mathbf{q})$$
 (3.14)

The calculation of the long-time behaviour of physical observables will be based on eqs. (3.11),(3.13), respectively. To do this in practice, g(t) must be found from the constraint (3.10), with the explicit form (3.12) of f(t) taken into account.

3.2 Linear responses

Since the evolution of the model starts from a non-stationary initial state and the dynamics contains an external force which does not derive from a Hamiltonian, it leads to non-equilibrium relaxation, where responses and correlators must be analysed separately. We begin with the response functions.

As the Arcetri model describes the growth of an interface, the conjugate variable to the interface height \hat{h} is an additional particle-deposition rate $\hat{\jmath} = \hat{\jmath}(s, \boldsymbol{q})$ at time s and at momentum \boldsymbol{q} . Such an extra term can simply be added to the equations of motions by formally replacing $\eta \mapsto \eta + j$. We then define the linear response of the interface height

$$\widehat{R}(t, s; \boldsymbol{p}, \boldsymbol{q}) := \left. \frac{\delta \langle \widehat{h}(t, \boldsymbol{p}) \rangle}{\delta \widehat{\jmath}(s, -\boldsymbol{q})} \right|_{i=0} = \Theta(t - s) \delta(\boldsymbol{p} - \boldsymbol{q}) \sqrt{\frac{g(s)}{g(t)}} e^{-2\Gamma(t - s)\omega(\boldsymbol{p})}$$
(3.15)

and used (3.13). For a linear response as considered here, g(t) is still given by (3.10). The Heaviside function $\Theta(t-s)$ expresses the causality condition t>s. In direct space, this gives

$$R(t, s; \mathbf{r} - \mathbf{r}') = \frac{\delta \langle h(t, \mathbf{r}) \rangle}{\delta j(s, \mathbf{r}')} \Big|_{j=0}$$

$$= \frac{\Theta(t-s)}{(2\pi)^d} \sqrt{\frac{g(s)}{g(t)}} \int_{\mathcal{B}} d\mathbf{p} \, e^{i\mathbf{p} \cdot (\mathbf{r} - \mathbf{r}') - 2\Gamma(t-s)\omega(\mathbf{p})} = \Theta(t-s) \sqrt{\frac{g(s)}{g(t)}} \, F_{\mathbf{r}}(t-s) \quad (3.16)$$

where here and in what follows we use the abbreviation (with the modified Bessel function I_n taken from [89, eq. (9.6.19)])

$$F_{\boldsymbol{r}}(\tau) := \frac{1}{(2\pi)^d} \int_{\mathcal{B}} d\boldsymbol{p} \cos(\boldsymbol{p} \cdot \boldsymbol{r}) e^{-2\Gamma\tau \omega(\boldsymbol{p})} = \prod_{a=1}^d e^{-2\Gamma\tau} I_{r_a}(2\Gamma\tau)$$
(3.17)

In order to derive the response of the slopes, consider first a source term in an action, expressed in momentum space

$$d\int d\boldsymbol{p}\,\widehat{h}(t,\boldsymbol{p})\widehat{\jmath}(t,-\boldsymbol{p}) = \int d\boldsymbol{p}\,\sum_{a=1}^{d}\widehat{h}(t,\boldsymbol{p})\frac{\mathrm{i}\,\sin p_{a}}{\mathrm{i}\,\sin p_{a}}\,\widehat{\jmath}(t,-\boldsymbol{p}) = \sum_{a=1}^{d}\int d\boldsymbol{p}\,\widehat{u}_{a}(t,\boldsymbol{p})\widehat{J}_{a}(t,-\boldsymbol{p})$$

using (3.4) and where we introduced the integrated source $\widehat{J}_a(t, \mathbf{p}) := (-i \sin p_a)^{-1} \widehat{\jmath}(t, \mathbf{p})$. Consider the following linear response

$$\widehat{Q}(t,s;\boldsymbol{p},\boldsymbol{q}) := \sum_{a=1}^{d} \frac{\delta \langle \widehat{u}_{a}(t,\boldsymbol{p}) \rangle}{\delta \widehat{J}_{a}(s,-\boldsymbol{q})} \bigg|_{j=0} = \delta(\boldsymbol{p}-\boldsymbol{q})\Theta(t-s)\sqrt{\frac{g(s)}{g(t)}} \sum_{a=1}^{d} \sin^{2}p_{a} e^{-2\Gamma(t-s)\omega(\boldsymbol{p})}$$
(3.18)

which in direct space leads to

$$Q(t, s; \boldsymbol{r} - \boldsymbol{r}') = \frac{1}{(2\pi)^{2d}} \int_{\mathcal{B}^2} d\boldsymbol{p} \, d\boldsymbol{q} \, e^{i\boldsymbol{p}\cdot(\boldsymbol{r} - \boldsymbol{r}')} \widehat{Q}(t, s; \boldsymbol{p}, \boldsymbol{q})$$

$$= \frac{\Theta(t - s)}{(2\pi)^d} \sqrt{\frac{g(s)}{g(t)}} \int_{\mathcal{B}} d\boldsymbol{p} \, \lambda(\boldsymbol{p}) \, e^{i\boldsymbol{p}\cdot(\boldsymbol{r} - \boldsymbol{r}') - 2\Gamma(t - s)\omega(\boldsymbol{p})}$$
(3.19)

In particular, the autoresponse of the slopes takes the simple form $Q(t,s) = Q(t,s;\mathbf{0}) = \Theta(t-s)f((t-s)/2)\sqrt{g(s)/g(t)}$.

3.3 Correlation functions

In order to write down the fluctuations and correlations of the height, we average the solution (3.13) for $p \neq 0$. Then

$$\langle \hat{h}(t, \mathbf{p}) \rangle = \langle \hat{h}(0, \mathbf{p}) \rangle \frac{\exp(-2\Gamma t\omega(\mathbf{p}))}{\sqrt{g(t)}}$$
 (3.20)

Going back to discrete momenta formulation for clarity, we have the deviation in the height

$$h(t, \mathbf{r}) - \langle h(t, \mathbf{r}) \rangle = \frac{1}{N^d} \sum_{\mathbf{p}} \left[\left(\widehat{h}(0, \mathbf{p}) - \langle \widehat{h}(0, \mathbf{p}) \rangle \right) \exp\left(\frac{2\pi i}{N} \mathbf{p} \cdot \mathbf{r} - 2\Gamma t \omega(\mathbf{p}) \right) \frac{1}{\sqrt{g(t)}} + \int_0^t d\tau \, \widehat{\eta}(\tau, \mathbf{p}) \exp\left(\frac{2\pi i}{N} \mathbf{p} \cdot \mathbf{r} - 2\Gamma(t - \tau)\omega(\mathbf{p}) \right) \sqrt{\frac{g(\tau)}{g(t)}} \right]$$
(3.21)

such that the two-time height-height correlator becomes

$$C(t, s; \mathbf{r} - \mathbf{r}') = C(s, t; \mathbf{r}' - \mathbf{r}) = \langle (h(t, \mathbf{r}) - \langle h(t, \mathbf{r}) \rangle) (h(s, \mathbf{r}') - \langle h(s, \mathbf{r}') \rangle) \rangle$$

$$= \frac{1}{N^d} \frac{1}{\sqrt{g(t)g(s)}} \sum_{\mathbf{p} \neq \mathbf{0}} \left[\langle \hat{h}(0, \mathbf{p}) \hat{h}(0, -\mathbf{p}) \rangle - \langle \hat{h}(0, \mathbf{p}) \rangle^2 \right] e^{2\pi i N^{-1} \mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')} e^{-2\Gamma(t+s)\omega(\mathbf{p})}$$

$$+ \frac{2\Gamma T}{N^d} \sum_{\mathbf{p}} \int_0^{\min(t, s)} d\tau \frac{g(\tau)}{\sqrt{g(t)g(s)}} e^{2\pi i N^{-1} \mathbf{p} \cdot (\mathbf{r} - \mathbf{r}')} e^{-2\Gamma(t+s-2\tau)\omega(\mathbf{p})}$$
(3.22)

For a vanishing initial width, see eq. (3.5) with $H_1 = H_0^2$, the term in the square brackets in the $2^{\rm nd}$ line vanishes for p = 0. We can now take again the $N \to \infty$ limit. Using $F_r(t)$ as defined in eq. (3.17), the two-time correlator reads (simplified for $t \ge s$)

$$C(t,s;\mathbf{r}) = \frac{H_1}{\sqrt{g(t)g(s)}} F_{\mathbf{r}}(t+s) + \frac{2\Gamma T}{\sqrt{g(t)g(s)}} \int_0^s d\tau \ g(\tau) F_{\mathbf{r}}(t+s-2\tau)$$
(3.23)

and the autocorrelator becomes $C(t,s) = C(t,s;\mathbf{0})$. In particular, the interface width reads

$$w^{2}(t) = C(t,t) = \frac{H_{1}F_{0}(2t)}{g(t)} + 2\Gamma T \int_{0}^{t} d\tau \, \frac{g(\tau)}{g(t)} F_{0}(2t - 2\tau)$$
(3.24)

Similarly, the slope-slope correlator can be found, again with $t \geq s$ assumed

$$\langle u_a(t, \mathbf{r}) u_a(s, \mathbf{r}') \rangle = \frac{H_1}{\sqrt{g(t)g(s)}} \frac{1}{(2\pi)^d} \int_{\mathcal{B}} d\mathbf{p} \sin^2 p_a \, e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')-2\Gamma(t+s)\omega(\mathbf{p})}$$

$$+ \frac{2\Gamma T}{(2\pi)^d} \int_{\mathcal{B}} d\mathbf{p} \int_0^s d\tau \, \frac{g(\tau)}{\sqrt{g(t)g(s)}} \sin^2 p_a \, e^{i\mathbf{p}\cdot(\mathbf{r}-\mathbf{r}')-2\Gamma(t+s-2\tau)\omega(\mathbf{p})}$$
(3.25)

whereas $\langle u_a u_b \rangle = 0$ if $a \neq b$. We shall be interested in particular in the autocorrelator of the slopes, with f(t) given by eq. (3.12)

$$A(t,s) := \sum_{a=1}^{d} \langle u_a(t, \mathbf{r}) u_a(s, \mathbf{r}) \rangle = H_1 \frac{f((t+s)/2)}{\sqrt{g(t)g(s)}} + 2\Gamma T \int_0^s d\tau \, \frac{g(\tau)}{\sqrt{g(t)g(s)}} \, f\left(\frac{t+s}{2} - \tau\right)$$
(3.26)

Here, and in what follows, we shall always choose units such that $\Gamma = 1$.

3.4 Equivalence with the spherical spin glass

As we now show, the 1D Arcetri model is equivalent to the kinetic p = 2 spherical spin glass, analysed by Cugliandolo and Dean [64].¹⁴

The spin hamiltonian is $\mathcal{H} = -\frac{1}{2} \sum_{i \neq j} J_{i,j} s_i s_j$, where $s_i \in \mathbb{R}$ are spherical spins which obey the spherical constraint $\sum_{i=1}^{N} s_i^2 = N$. The elements of the symmetric matrix J are independent gaussian random variables with zero mean and variance proportional to 1/N. Then the thermodynamic limit is well-defined and the probability distribution of the eigenvalues μ of J is given by the Wigner semi-circle law

$$\rho(\mu) = \frac{\sqrt{4 - \mu^2}}{2\pi} \tag{3.27}$$

The corresponding eigenvectors of a spin configuration $s(t) = \{s_i(t)\}$ are denoted by $s_{\mu}(t) = \mu \cdot s(t)$. Consider uniform initial conditions $s_{\mu}(0) = 1$. The time-evolution is given by the Langevin equation [64, eq. (2.2)]

$$\partial_t s_{\mu}(t) = (\mu + \mathfrak{z}(t)) \, s_{\mu}(t) + h_{\mu}(t) + \xi_{\mu}(t) \tag{3.28}$$

where $h_{\mu}(t)$ is an external magnetic field and $\xi_{\mu}(t)$ is the centered thermal noise, with variance $\langle \xi_{\mu}(t) \xi_{\nu}(t') \rangle = 2T_{\rm SG}\delta_{\mu\nu}\delta(t-t')$. The solution of the Langevin equation is now immediate. Defining $\gamma(t) := \exp(-2\int_0^t d\tau \,\mathfrak{z}(\tau))$, the spherical constraint can be rewritten as a Volterra integral equation [64, eq. (2.7)]

$$\gamma(t) = \langle \langle s_{\mu}(0) \exp(2\mu t) \rangle \rangle + 2T_{\text{SG}} \int_{0}^{t} d\tau \, \gamma(\tau) \, \langle \langle \exp 2\mu(t-\tau) \rangle \rangle$$
 (3.29)

where the following average over the Wigner distribution $\rho(\mu)$ is carried out [64]

$$\langle \langle \exp(2\mu t) \rangle \rangle := \int_{2}^{2} d\mu \, \rho(\mu) e^{2\mu t} = \frac{2}{\pi} \int_{1}^{1} d\mu \, \sqrt{1 - \mu^{2}} \, e^{4\mu t} = \frac{I_{1}(4t)}{2t}$$
 (3.30)

where [89, eq. (9.6.18)] was used.

In order to see the relationship of these results with the 1D Arcetri model, we first observe that the eigenvalue spectrum in the spherical spin glass is in the interval $\mu \in [-2, 2]$, whereas the dispersion relation of the Arcetri model $2\omega(p) \in [0, 4]$. This can be matched through the

 $^{^{14}}$ A careful analysis [73, 90] shows that the ageing properties of truly glassy systems on one hand, and simple spin systems and the p=2 spherical spin glass on the other hand, are quite distinct.

mapping $\mu \mapsto -2 + \mu$ and $\mathfrak{z}(t) \mapsto \mathfrak{z}(t) + 2$ in order to keep the equation of motion (3.28) unchanged. Identifying the relationship

$$g(t) = e^{-4t}\gamma(t) \tag{3.31}$$

it follows that the Volterra equation (3.29) with the explicit average (3.30) becomes exactly the spherical constraint (3.10), where f(t) is given in (3.12) and the identifications

$$T = 2T_{SG}$$
 and $H_1 = 2$ (3.32)

Next, the magnetic autoresponse of the spin glass (with t > s assumed) [64, eqs. (2.16,3.17)]

$$R_{\rm SG}(t,s) := \sum_{i=1}^{N} \left. \frac{\delta \langle s_i(t) \rangle}{\delta h_i(s)} \right|_{h=0} = \sqrt{\frac{\gamma(s)}{\gamma(t)}} \left. \langle \langle \exp \mu(t-s) \rangle \rangle \right| = 2\sqrt{\frac{g(s)}{g(t)}} f\left(\frac{t-s}{2}\right) = 2Q(t,s)$$
(3.33)

is identical to the slope autoresponse (3.19), up to a factor 2. Finally, and recalling (3.30), the disorder-averaged spin glass autocorrelator [64, eqs. (2.12,3.6)]

$$C_{SG}(t,s) := \frac{1}{N} \left[\sum_{i=1}^{N} \langle s_i(t) s_i(s) \rangle \right]_J = \int_{-2}^{2} d\mu \, \rho(\mu) \, \langle s_\mu(t) s_\mu(s) \rangle$$

$$= \frac{1}{\sqrt{\gamma(t)\gamma(s)}} \left(\langle \langle \exp \mu(t+s) \rangle \rangle + 2T_{SG} \int_{0}^{\min(t,s)} d\tau \, \gamma(\tau) \, \langle \langle \exp \mu(t+s-2\tau) \rangle \rangle \right) = A(t,s)$$
(3.34)

reduces to the slope-slope autocorrelator (3.26), with the same identifications (3.32) as above.

Since it is well-known that the spherical spin glass is in the same universality class as the 3D kinetic spherical model [64, 65], it follows that the 1D Arcetri model is in the same universality class as well. The physical correspondence is between the slopes in the Arcetri model and the magnetic spherical spins, see (3.33),(3.34). This can be generalised: the long-time asymptotics of g(t) and f(t) imply that the exponents which give the long-time behaviour of A(t,s) and Q(t,s) of the d-dimensional Arcetri model are the same as those of the magnetic autocorrelators and autoresponses in the kinetic spherical model [65] in d+2 dimensions.

4 Long-time behaviour

Given the form of the spherical constraint (3.10) as a Volterra integral equation, its solution proceeds via standard Laplace transforms [61, 64, 65]. Let $\overline{g}(p) = \mathcal{L}(g(t))(p) = \int_0^\infty dt \ g(t)e^{-pt}$ denote the Laplace transform. Then (3.10) can be inverted

$$\overline{g}(p) = \frac{H_1 \overline{f}(p)}{d - 2T \overline{f}(p)} \tag{4.1}$$

Standard Tauberian theorems [109, ch. XIII.5] relate the long-time behaviour of g(t) to the $p \to 0$ behaviour of $\overline{g}(p)$. Hence, expanding $\overline{f}(p)$ for $p \ll 1$, one can find first $\overline{g}(p)$ and then g(t), for t large enough. The explicit calculations are detailed in appendix A.

4.1 Fast relaxation for $T > T_c$

Consider the situation when the denominator in (4.1) vanishes for some value $p_0 > 0$: $d - 2T\overline{f}(p_0) = 0$. Since $\overline{f}(p)$ decreases monotonously with p, this zero exists and is simple for T large enough. Hence $\overline{g}(p)$ has a simple pole at $p = p_0$. Transforming back, one has

$$g(t) \stackrel{t \to \infty}{\simeq} -\frac{H_1}{2T} \frac{\overline{f}(p_0)}{\overline{f}'(p_0)} e^{p_0 t} = -\frac{H_1 d}{4T^2} \frac{1}{\overline{f}'(p_0)} e^{p_0 t}$$
(4.2)

Here $1/p_0$ is the finite relaxation time and governs the time-translation-invariant approach towards the stationary state ($w_{0,1}$ are constants)

$$R(s+\tau, s; \mathbf{r}) \sim \tau^{-d/2} \exp\left(-\frac{1}{2}p_0\tau - \frac{\mathbf{r}^2}{4\tau}\right)$$

$$w^2(t) \sim w_0 T + w_1 e^{-p_0 t}$$

$$C(s+\tau, s) \sim \exp\left(-\frac{1}{2}p_0\tau\right)$$

$$(4.3)$$

This rapid relaxation does not show dynamical scaling and is of no particular interest to us.

4.2 Ageing at the critical point

The smallest temperature $T = T_c$, for which $\overline{g}(p)$ has a singularity, occurs when $d/(2T_c) = \overline{f}(0)$. Hence the *critical temperature* $T_c = T_c(d)$ is given by

$$\frac{1}{T_c(d)} = \frac{2}{d}\overline{f}(0) = \frac{1}{2} \int_0^\infty dt \, e^{-dt} t^{-1} I_1(t) I_0(t)^{d-1}$$
(4.4)

Clearly, $T_c(d) > 0$ for all d > 0. Using [89, eq. (11.4.13)] and [91, eq. (2.15.20.6)], we have the explicit values¹⁵

$$T_c(1) = 2$$
 , $T_c(2) = \frac{2\pi}{\pi - 2} \simeq 5.5038...$, $T_c(3) \simeq 9.53099...$ (4.5)

In addition, we have $T_c(d) \simeq d$ for $d \ll 1$ and $T_c(d) \simeq 4d$ for $d \gg 1$.

We now write down the linear response (3.16), the interface width (3.24) and the autocorrelator (3.23), in the long-time scaling limit, where both $t, s \to \infty$ such that y = t/s is kept fixed. We begin with the two-time time-space response and expand g(t) and $F_r(t)$ for large times. This gives

$$R(t,s;\mathbf{r}) = \sqrt{\frac{g(s)}{g(t)}} F_{\mathbf{r}}(t-s) \simeq R(t,s) \exp\left[-\frac{1}{4}\frac{\mathbf{r}^2}{t-s}\right]$$

$$R(t,s) = s^{-d/2} f_R(t/s)$$
(4.6)

with the explicit scaling function

$$f_R(y) = (4\pi)^{-d/2} (y-1)^{-d/2} y^{(2-d)/4}; \text{ if } 0 < d < 2$$

 $f_R(y) = (4\pi)^{-d/2} (y-1)^{-d/2}; \text{ if } 2 < d$ (4.7)

¹⁵For d=1 in agreement with the spin glass critical point $T_{SG,c}=1$ [64].

From this, the values of the non-equilibrium exponents a, λ_R and also the dynamical exponent z=2 can be read off and are listed in table 1.

Next, consider the two-time correlator. In the scaling limit, with t = ys > s, and for 0 < d < 2, we find

$$C(t, s; \mathbf{r}) \simeq 2T_c (ts)^{(2-d)/4} \int_0^s d\tau \, \tau^{d/2-1} F_{\mathbf{r}}(t+s-2\tau)$$

$$\stackrel{s \to \infty}{\simeq} s^{1-d/2} \, 2T_c \, y^{(2-d)/4} \int_0^1 du \, u^{d/2-1} \left(4\pi \left[(y+1) - 2u \right] \right)^{-d/2} \exp \left[-\frac{\mathbf{r}^2}{s} \frac{1}{y+1-2u} \right]$$
(4.8)

which is of the expected scaling form $C(t, s; \mathbf{r}) = s^{-b} F_C(t/s, \mathbf{r}^2/s)$. For d > 2, an analogous calculation gives

$$C(t, s; \mathbf{r}) = s^{1-d/2} 2T_c \int_0^1 du \, u^{d/2-1} \left(4\pi \left[(y+1) - 2u \right] \right)^{-d/2} \exp \left[-\frac{\mathbf{r}^2}{s} \frac{1}{y+1-2u} \right]$$
(4.9)

Especially, the autocorrelator $C(t,s) = C(t,s;\mathbf{0}) = s^{-b}F_C(t,s;\mathbf{0}) = s^{-b}f_C(t/s)$ has the explicit scaling function ¹⁶

$$f_C(y) = \frac{4T_c(d)}{d(4\pi)^{d/2}} \frac{y^{(2-d)/4}}{(1+y)^{d/2}} {}_2F_1\left(\frac{d}{2}, \frac{d}{2}; 1 + \frac{d}{2}; \frac{2}{1+y}\right) \sim y^{-(3d-2)/4} ; \text{ if } 0 < d < 2$$

$$f_C(y) = \frac{4T_c(d)}{d(4\pi)^{d/2}} (1+y)^{-d/2} {}_2F_1\left(\frac{d}{2}, \frac{d}{2}; 1 + \frac{d}{2}; \frac{2}{1+y}\right) \sim y^{-d/2} ; \text{ if } 2 < d \qquad (4.10)$$

along with the asymptotics for y large. The values of the exponents b and λ_C are listed in table 1.

In analogy with the known results of the kinetic spherical model of magnets [74, 79], the behaviour at the upper critical dimension $d = d^* = 2$ of the two-time observables $R(t, s; \mathbf{r})$ and $C(t, s; \mathbf{r})$ can be obtained by formally taking the limit $d \to 2$ in eqs. (4.7) and (4.10). Hence no logarithmic modifications of the leading scaling behaviour arise, but there are additive logarithmic corrections to scaling. This is different for single-time quantities like the interface width w(t), of which the leading long-time behaviour is

$$w^{2}(t) = \frac{2\pi T_{c}(d)}{(8\pi)^{d/2} \sin(\pi d/2)} t^{1-d/2} ; \text{ if } 0 < d < 2$$

$$w^{2}(t) \sim 2T_{c} \ln t ; \text{ if } d = 2$$

$$w^{2}(t) = 2T_{c}(d) \mathcal{F}_{d} + O(t^{1-d/2}) ; \text{ if } d > 2$$

$$(4.11)$$

with the constant $\mathcal{F}_d := \int_0^\infty d\tau \ (e^{-4\tau}I_0(4\tau))^d$, which diverges for $d \searrow 2$.¹⁷ Hence, the interface width $w(t) \sim \sqrt{\ln t}$ growths logarithmically in 2D, see table 1, in agreement also with experimental findings [40]. The values of exponent β , listed in table 1, are the same as for the EW-universality class.

Inspection of the values of the various exponents collected in table 1 shows that

Recast the integral $\int_0^1 du \, u^{d/2-1} \left[(y+1) - 2u \right]^{-d/2} = \frac{2}{d} (1+y)^{-d/2} {}_2F_1 \left(d/2, d/2; 1 + d/2; 2/(1+y) \right)$ = $(y-1)^{-d/2} B_{2/(1+y)} \left(d/2, 1 - d/2 \right)$ as a hypergeometric or incomplete Beta function [89, eqs.(15.3.1,26.5.23)]. $^{17} \mathcal{F}_3 = \pi^{-2} \left(18 + 12\sqrt{2} - 10\sqrt{3} - 7\sqrt{6} \right) \mathbf{K}^2 \left[(2-\sqrt{3})(\sqrt{3}-\sqrt{2}) \right] \simeq 0.1263655...$, where $\mathbf{K}(k)$ is the complete elliptic integral [91, eq.(2.15.21.2)].

- 1. the stationary exponents z, β , and consequently also the roughness exponent $\alpha = z\beta$, of the Arcetri model are in all dimensions the same as those of the Edwards-Wilkinson model. This might have been anticipated, since the equation of motion (3.2) is still linear and also reflects properties of the magnetic spherical model, where the equilibrium critical exponent $\beta = \frac{1}{2}$ of the order parameter and $\eta = 0$ of the spin-spin correlator keep their mean-field values [59].
- 2. the non-equilibrium autocorrelation/autoresponse exponents $\lambda_C = \lambda_R$ depend on the dimensionality in a non-trivial way. Only for $d \geq d^* = 2$, their values are the same as in the EW-universality class.

In this respect, the dynamics of the Arcetri model is quite similar to what is found in the non-equilibrium critical dynamics of *simple magnets* with a non-conserved order parameter and disordered initial states. For such systems, a long-standing result of field-theory [13, 21, 7] asserts that the true non-equilibrium exponents λ_C , λ_R should be independent of the stationary exponents (since an independent renormalisation is required for their calculation [13]). The rôle of the stationary exponents is taken here by z, α , β and the exponents a, b related to them. Hence the Arcetri model with d < 2 can be considered as an explicit example of this general fact.

We also recall that in the KPZ class with d < 2, a Ward identity prevents this additional renormalisation, to all orders in perturbation theory [22], which is a qualitatively different situation. We are not aware of any estimate of λ_C from a non-perturbative renormalisation-group study in the KPZ universality class, for $d \geq 2$.

- 3. for the KPZ-universality class, there is a conjecture, ¹⁸ for a flat interface, that $\lambda_C \stackrel{!}{=} d$ [14]; this also happens to be satisfied in the EW-model. However, this conjecture does not extend to the Arcetri model, since $\lambda_C = \lambda_R \neq d$ for d < 2.
- 4. in the Arcetri model, autocorrelation and autoresponse exponents are always equal. Is the available numerical evidence for $\lambda_C \neq \lambda_R$ in the 2D KPZ-model [27] the final word?

Remarkably, even for d > 2, the Arcetri and the EW models are in different universality classes, in spite of all their exponents being equal. In order to see this, recall that because of a = b, one can define the fluctuation-dissipation ratio (FDR) [63] (for reviews, see [92, 21, 93, 94, 95])

$$X(t,s) := TR(t,s) \left(\frac{\partial C(t,s)}{\partial s}\right)^{-1} = X\left(\frac{t}{s}\right)$$
(4.12)

where the last relation holds in the scaling limit. In magnetic systems or spin glasses, one uses the value of X(y) - 1 as a measure of the distance with the respect to an equilibrium state. According to the Godrèche-Luck conjecture [65], the limit FDR $X_{\infty} = \lim_{y \to \infty} X(y)$ should be an universal number. For the critical Arcetri model, the explicit results (4.7), (4.10) lead to

$$X_{\infty} = \begin{cases} d/(d+2) & ; & \text{if } 0 < d < 2\\ d/4 & ; & \text{if } d \ge 2 \end{cases}$$
 (4.13)

¹⁸For the KPZ class in d < 2 dimensions, $\lambda_C = d$ has been proven to all orders in perturbation theory [22].

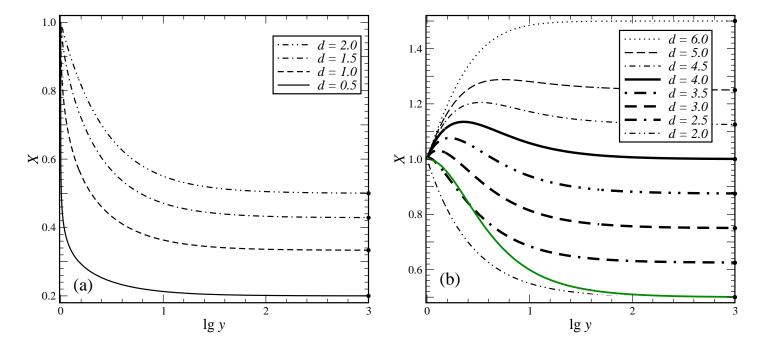


Figure 2: Fluctuation-dissipation ratio X(y) over against y = t/s of the critical Arcetri model for (a) $d \le 2$ and (b) $d \ge 2$. The small circles on the right axis indicate the limit ratio X_{∞} , eq. (4.13). In (b), the solid green line gives X(y) for the EW-model in d = 4 dimensions.

Eq. (4.13) is distinct from the well-known value $X_{\infty}^{(\mathrm{EW})} = \frac{1}{2}$ of the EW-model, valid in any dimension [15].¹⁹ To illustrate this further, we show in figure 2 the scaling function X(y) as a function of y = t/s, for d below (figure 2a) and above (figure 2b) the upper critical dimension $d^* = 2$. Starting from the value X(1) = 1 at equal times, the limit ratio X_{∞} is continuously approached for an increasing temporal separation $y = t/s \to \infty$. Qualitatively, this approach is monotonous for dimensions $d \leq 2$, but for d > 2 but not too large, X(y) goes through a maximum before approaching its limit value. For sufficiently small dimensions, this approach is from above, but when d becomes large enough, the maximum of X(y) disappears and the limit value X_{∞} is approached monotonously from below. Curiously, for d = 4 one has $X_{\infty} = 1$.

For d=2, the FDR of the EW-model is identical to the Arcetri model and the green solid line in figure 2b gives the FDR of the EW-model for d=4. For all dimensions 2 < d < 4, the FDRs of the EW-model fall between these two curves, which are quite distinct from those of the Arcetri model.

The distinction between the Arcetri and EW-models is further illustrated in figure 3, where the ratio of the two fluctuation-dissipation ratios is plotted. With an increasing separation y = t/s > 1 of the time-scales the distinction of the two models becomes progressively more clear. Only for d = 2 the two functions X(y) are identical, hence the two models are indeed identical in 2D.²⁰

Hence the Arcetri model with $d \geq 2$ illustrates the usefulness of the fluctuation-dissipation

¹⁹The slope autocorrelator A(t,s) and autoresponse Q(t,s) of the d-dimensional Arcetri model have the same scaling behaviour as the magnetic autocorrelator and autoresponse in the (d+2)-dimensional spherical model, see sec. 3.4. For d < 2, the limit 'slope FDR' $X_{\infty}^{\text{SM}}(d+2) = 1 - 2/(d+2) = d/(d+2)$ [65] agrees with the height FDR (4.13) of the d-dimensional Arcetri model.

²⁰Therefore, the experiments described in [40] apply to both.

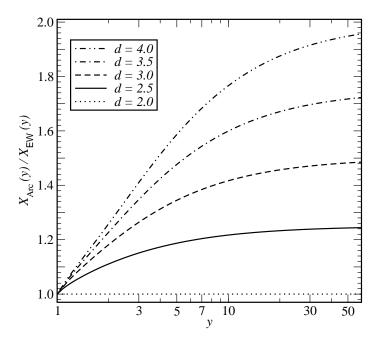


Figure 3: Ratio of the fluctuation-dissipation ratio $X_{Arc}(y)$ for the Arcetri model and of $X_{EW}(y)$ for the EW-model, as a function of y = t/s, for several dimensions $2 \le d \le 4$.

ratio as a diagnostic tool for a fine distinction of non-equilibrium universality classes. It also points to the interest of measuring experimentally both two-time correlators and responses in growing interfaces (long-range interactions generically reduce the value of d^*).

4.3 Relationship with the bosonic pair-contact process

Surprisingly, it turns out that for 2 < d < 4, the non-equilibrium relaxation properties of the critical Arcetri model are related to those of a different stochastic process, the so-called (multi-)critical bosonic pair-contact process with diffusion (BPCPD) [96, 97, 98], see [99] for a field-theoretic description. In this model, each site of a d-dimensional hypercubic lattice can be occupied by an arbitrary number $n \in \mathbb{N}$ of particles of a single species A. Single particles may hop to a nearest-neighbour site, with a diffusion rate D. On the same site, pairs of particles may react, according to the schemes $2A \to (2+k)A$ or $2A \to (2-k)A$, where k is either one or two. The universal long-time behaviour of the model does not depend on the value of k. The two reaction rates are chosen equal $\Gamma[2A \to (2+k)A] = \Gamma[2A \to (2-k)A] = \mu$. The model's behaviour is determined by the value of the control parameter $\alpha := k^2\mu/D$. There is a critical value, $\alpha = \alpha_C$, with²¹

$$\frac{1}{\alpha_C} = 2 \int_0^\infty du \left(e^{-4u} I_0(4u) \right)^d \tag{4.14}$$

and for d > 2, one has $\alpha_C > 0$. For $\alpha > \alpha_C$, all particles cluster on a few sites only [97, 98, 9] while for $\alpha < \alpha_C$, the system evolves towards a spatially homogeneous stationary state. We are interested in the multi-critical point at $\alpha = \alpha_C > 0$, where this 'clustering transition' occurs. For $d \leq 2$, the multi-critical point in the BPCPD does not exist.

²¹Using \mathcal{F}_3 from (4.11), $\alpha_C = (2\mathcal{F}_3)^{-1} \simeq 3.956776...$ for d = 3.

The dynamics of the model is described in terms of the bosonic particle-annihilation operator $a(t, \mathbf{r})$ such that the particle-number operator is $n(t, \mathbf{r}) = a^{\dagger}(t, \mathbf{r})a(t, \mathbf{r})$. For any value of α , and an initial state with uncorrelated particles of a mean number density ρ_0 , the average particle-number $\langle n(t, \mathbf{r}) \rangle = \langle a(t, \mathbf{r}) \rangle = \rho_0$ is constant [97, 98]. The two-time density-density correlator can be expressed as [9]

$$\langle n(t,\mathbf{0})n(s,\mathbf{r})\rangle - \rho_0^2 = \bar{C}(t,s;\mathbf{r}) + \bar{R}(t,s;\mathbf{r})\rho_0$$
(4.15)

with the following definition of the two-time correlators and responses [98]

$$\bar{C}(t,s;\boldsymbol{r}) = \left\langle a^{\dagger}(t,\boldsymbol{0})a(s,\boldsymbol{r})\right\rangle - \rho_0^2 , \quad \bar{R}(t,s;\boldsymbol{r}) = \left. \frac{\delta \left\langle a(t,\boldsymbol{0})\right\rangle}{\delta j(s,\boldsymbol{r})} \right|_{i=0}$$
(4.16)

where j denotes the particle-creation rate conjugate to the particle number. The usual scaling behaviour (1.2),(1.3) has been confirmed, for all $\alpha \leq \alpha_C$. Precisely at the clustering transition $\alpha = \alpha_C$, and for 2 < d < 4 dimensions, the explicit scaling behaviour reads, for $s \to \infty$ [98]

$$\bar{C}(ys,s;\mathbf{0}) = c_0 \rho_0^2 (y+1)^{-d/2} {}_2F_1\left(\frac{d}{2},\frac{d}{2};1+\frac{d}{2};\frac{2}{1+y}\right) , \quad \bar{R}(ys,s;\mathbf{0}) = s^{-d/2} r_0 (y-1)^{-d/2}$$

$$(4.17)$$

and where c_0, r_0 are known normalisation constants. Hence, $\bar{C}(t, s; \mathbf{0})$ can be considered as the scaling limit of the connected density-density autocorrelator in the BPCPD. For 2 < d < 4, the shape of the scaling functions is identical to the ones of the Arcetri model given in eqs. (4.10),(4.7). However, although a = d/2 - 1 in both models, the values of the ageing exponents are distinct, namely b = 0 for the BPCPD and b = d/2 - 1 for the Arcetri model. Since $a \neq b$, the fluctuation-dissipation ratio (4.12) cannot be defined in the multi-critical BPCPD.

This is a further example which illustrates that the values of the non-equilibrium exponents $\lambda_{C,R}$ are independent from the stationary exponents α, β, z , analogously to non-equilibrium field theory of critical magnets [13, 21, 7], but different from the KPZ universality class [22, 7]. However, this independence is illustrated in the Arcetri model for 2 < d < 4 in a new way. Recall that for d < 2, all the exponents α, β, z and a, b are found to be the same in the EW and Arcetri universality classes, yet the values of $\lambda_C = \lambda_R$ are different. Here, when 2 < d < 4, the values of the non-equilibrium exponents $\lambda_C = \lambda_R = d$, as as well the ageing exponent a = d/2 - 1, are the same. However, the ageing exponent b = 0 in the BPCPD and b = d/2 - 1 in the Arcetri model are different.

Finally, for d > 4, the shape of $f_C(y)$ in the multi-critical BPCPD is different from (4.17), although $\lambda_C = d$, and b = d/2 - 2 [98]. This gives a different example, but of the same kind, as found for d < 4.

We emphasise that although the value of the autocorrelation exponent $\lambda_C = d$, the two-time correlation functions are distinct from those of the EW-model, where $C(t,s) = s^{-b} f_C^{(\mathrm{EW})}(t/s)$ with b = d/2 - 1 and $f_C^{(\mathrm{EW})}(y) = \frac{2T}{2-d}(4\pi)^{-d/2} \left[(y+1)^{1-d/2} - |y-1|^{1-d/2} \right]$ [15]. This rather coïncides with the shape of $f_C(y)$ in the critical BPCPD with $\alpha < \alpha_C$ [98].

4.4 Ageing for $T < T_c$

In order to discuss the behaviour below the critical point, when $T < T_c$ and the noise is thus relatively weak, it is useful to introduce the auxiliary variable $m^2 := 1 - T/T_c$.

The two-time response function again takes the scaling form (4.6), where the scaling function $f_R(y)$ of the autoresponse now becomes

$$f_R(y) = (4\pi)^{-d/2} y^{(d+2)/4} (y-1)^{-d/2}$$
(4.18)

for all dimensions d > 0. Similarly, the two-time autocorrelator becomes (with t = ys > s assumed; the first correction is valid for d < 2, see appendix A)

$$C(t,s) = 2^{d+2}m^2s \, y^{(2+d)/4}(y+1)^{-d/2} + O\left(Ts^{1-d/2}\right)$$
(4.19)

which has the expected scaling form $C(t,s) = s^{-b} f_C(t/s)$ with the explicit scaling function

$$f_C(y) = 2^{d+2} m^2 y^{(2+d)/4} (y+1)^{-d/2}$$
(4.20)

The non-equilibrium exponents can be read off and are listed in table 1. The extension to timespace correlations $C(t, s; \mathbf{r})$ is straightforward. The interface width is obtained as the special case $w^2(t) = C(t, t)$ and reads (the first correction is valid for d < 2)

$$w^{2}(t) = 4m^{2}t + O\left(Tt^{1-d/2}\right) \tag{4.21}$$

For correlators and the interface width, the long-time behaviour only depends on the ratio T/T_c and the short-ranged nature of the initial correlations (parametrised by H_1). The 'thermal' convolution term, which dominated the critical case, merely gives rise to a correction to scaling. This leading result $w^2(t) \simeq 4m^2t$, valid for t large enough, is independent of d, hence the growth exponent $\beta = \frac{1}{2}$.

4.5 Global persistence

A different class of observables are first-passage probabilities, such as persistence probabilities, see [100] for a recent detailed review. The global persistence probability $P_{\rm G}(t)$ is the probability that a fluctuation $\delta h_{\rm G}(t)$ of the global height $h_{\rm G}(t) \sim \int_{\mathbb{R}^d} \mathrm{d} \boldsymbol{r} \, h(t,\boldsymbol{r}) \sim \hat{h}(t,\boldsymbol{0})$ did never change its sign between the initial instant t=0 and time t. For large times, one expects a power-law decay $P_{\rm G}(t) \sim t^{-\theta_{\rm G}}$, where $\theta_{\rm G}$ is the global persistence exponent. In the infinite-size limit, the central limit theorem implies that the global height should be a gaussian variable [101, 100]. Remarkably, it can be shown that then the model's underlying stochastic process is markovian if and only if the normalised global autocorrelator

$$\widehat{N}(t,s) := \frac{\widehat{C}_{\mathbf{0}}(t,s)}{\sqrt{\widehat{C}_{\mathbf{0}}(t,t)\,\widehat{C}_{\mathbf{0}}(s,s)}} = \left(\frac{s}{t}\right)^{\mu} \tag{4.22}$$

(where $\widehat{C}_{\mathbf{0}}(t,s)$ is the two-time autocorrelator in momentum space) takes in the scaling limit, $t,s\to\infty$ with y=t/s fixed, a simple power-law form [101]. Then $\theta_{\rm G}=\mu$, and in addition, a scaling relation relating $\theta_{\rm G}$ and the autocorrelation exponent λ_C can be derived, for quenches either to $T=T_c$ [101] or to $T< T_c$ [102, 103, 100]. Adapting this to the exponent notation (1.2),(1.3) used for growing interfaces, this 'markovian' scaling relation reads

$$\theta_{G} = \frac{1}{z} \left(\lambda_{C} - \frac{d}{2} - \frac{zb}{2} \right) \ge 0 \tag{4.23}$$

where the bound follows from a Yeung-Rao-Desai inequality [104], see appendix B. However, in many non-equilibrium systems, quenched to either $T = T_c$ or $T < T_c$, those scaling relations turn out to be broken (albeit by numerically small amounts) so that one should conclude that the underlying stochastic processes cannot be markovian at sufficiently large times, see [100, 9] and references therein for explicit examples. Practically, deviations of $\hat{N}(t, s)$ from a pure power law may be more easy and more reliable to detect than tiny deviations from (4.23) [103].

Applying this general method to the Arcetri model, with (3.13) the global height autocorrelator is $\widehat{C}_{\mathbf{0}}(t,s) = \left(H_1 + 2\Gamma T \int_0^s d\tau \, g(\tau)\right) / \sqrt{g(t)g(s)}$, for $t \geq s$. Recalling the leading long-time behaviour of g(t) from appendix A, we have

$$\widehat{N}(t,s) = \sqrt{\frac{H_1 + 2\Gamma T \int_0^s d\tau \, g(\tau)}{H_1 + 2\Gamma T \int_0^t d\tau \, g(\tau)}} \stackrel{t,s \to \infty}{=} \left(\frac{s}{t}\right)^{\mu} , \quad \mu = \theta_{\rm G} = \begin{cases} 0 & ; & \text{if } T < T_c \text{ and } d > 0\\ d/4 & ; & \text{if } T = T_c \text{ and } 0 < d < 2\\ 1/2 & ; & \text{if } T = T_c \text{ and } d > 2 \end{cases}$$
(4.24)

As expected, the verification of the simple power-law (4.22) confirms the Markov property, hence the identification $\mu = \theta_G$ is admissible.

5 Discussion and perspectives

Concluendum est: this work explores the properties of an analogue of the familiar spherical model of ferromagnets [59] for the description of growing interfaces. In the scheme presented here, we admitted that the correspondence should be made between the local slopes of the interface and the spherical spins. It then becomes relatively straightforward to write down and solve the Langevin equations of motion, since essentially all techniques can be borrowed from the kinetic spherical model [61, 64, 65]. In particular, we have analysed the long-time behaviour of the Arcetri model, defined in section 2. Our findings are as follows:

- 1. the parameter T, named a 'temperature' by analogy with the kinetic spherical model, and defined through the noise correlator (2.2), admits for all dimensions d > 0 a critical value $T_c = T_c(d)$, given by eq. (4.4). Qualitatively, the Arcetri model is in this respect more analogous to magnetic systems and qualitatively different from more usual interface growth universality classes, such as described by the KPZ or EW universality classes.
 - At criticality $T = T_c(d)$, starting from a flat substrate, the interface becomes rough for $d \leq 2$ and remains smooth for d > 2. For $T < T_c(d)$, the interface always becomes rough and for $T > T_c(d)$, even an initially rough interface becomes smooth in a finite time.
- 2. for $T > T_c(d)$, the model's relaxation behaviour is rapid, time-translation-invariant and governed by a finite relaxation time.
- 3. in contrast, for $T \leq T_c$, the relaxation is slow, time-translation-invariance is broken and the two-time observables obey dynamical scaling (1.2),(1.3) for large times. Hence the conditions for *physical ageing*, as defined in [9], are obeyed.
 - Such a physical ageing behaviour has already been found in more standard models of growing interfaces, especially in the KPZ- and EW-models. In table 1, the values of the relevant exponents are listed.

- 4. the critical Arcetri model has an upper critical dimension $d^* = 2$.
 - If $d = d^* = 2$, no logarithmic modifications of the dynamical scaling arise for the twotime observables, but a logarithmic behaviour is seen for single-time quantities such as the interface width. The 2D Arcetri and EW-models are identical.
- 5. in many respects, if one chooses $T = T_c$, the resulting behaviour of the critical Arcetri model is qualitatively quite analogous to what has been found in other universality classes. However, some differences can be seen as well:
 - (a) the stationary exponents z, β, α (as well as the exponents a, b which depend on them), are the same for the EW-model and the Arcetri model. However, for d < 2, the non-equilibrium exponents λ_C and λ_R of both models are different. In this respect, the critical Arcetri model behaves analogously to the non-equilibrium critical dynamics of simple non-conserved magnets and is qualitatively different from growth models in the KPZ universality class [22]. It provides an explicit example where, for disordered initial states, the non-equilibrium exponents $\lambda_{C,R}$ are independent (i.e. not related by a scaling relation) from the stationary exponents. This is a long-standing prediction from the non-equilibrium critical dynamics of simple non-conserved magnets [13, 21, 7].
 - (b) in the Arcetri model, one has always $\lambda_C = \lambda_R$. This matches the exact results of the EW-model and the 1D KPZ-model. Curiously, the available numerical data indicate the contrary for the 2D KPZ-model [27].
 - (c) the conjecture $\lambda_C \stackrel{!}{=} d$ [22, 14], formulated for the KPZ equation, and observed to hold in the EW-model as well, does not extend to the Arcetri model when d < 2. The available numerical data for the 2D KPZ-model [27, 28] do not agree with it either. We remark that for the 1D KPZ equation, the shape of the two-time response function could only be explained using the logarithmic extension of local scale-invariance, which implies logarithmic terms in the scaling function [18, 105]. If such an observation could be extended to the 2D case as well: could un-recognised logarithmic contributions have modified the effective values of λ_C or λ_R ?
- 6. for dimensions 2 < d < 4, the two-time response function, and the shape of the scaling functions of the two-time correlator (up to normalisation), are the same as at the multi-critical point in the so-called bosonic pair-contact process with diffusion (BPCPD) [98]. However, although the asymptotic exponent has the same value $\lambda_C = d$, the shape of the scaling function of the two-time correlator of the EW-model is distinct from the one of the Arcetri model for d > 2, although the two-time responses $R(t, s; \mathbf{r})$ and the exponent b = d/2 1 agree.
- 7. an universal method to distinguish the Arcetri and EW-models for d>2 uses the fluctuation-dissipation ratio (4.12) [63], in particular the universal limit FDR X_{∞} [65] has different values in the two universality classes.²²
 - Hence, for d > 2, we have three distinct models, with the same values of the exponents a, and $\lambda_R = \lambda_C$, but which are distinguished as follows: (i) between the Arcetri and

²²The distinction between the two models for $d \neq 2$ should be related to the spherical constraint (2.4), which is incompatible with a flat surface $h_n = \text{cste.}$.

BPCPD-classes, the exponent b has different values. (ii) between the Arcetri and EW-classes, the FDR and in particular the limit X_{∞} are different. This illustrates once more the independence of the exponent λ_C from the other ones.

8. the relationship with the kinetic spherical model in d+2 dimensions (see section 3.4), relating local slopes to local magnetisations, allows to draw a clear physical picture on how the long-time relaxation is going on: for $T < T_c$, the Arcetri model undergoes a coarsening, such that increasingly larger patches of the interface (of typical size $L(t) \sim t^{1/2}$ for all d and $T \leq T_c$ [61, 79]) have constant slope, leading to a saw-tooth pattern. For $T = T_c$, the interface slopes are correlated over increasing distances L(t), such that the interface itself should become a fractal.

The spherical model is easily adapted and generalised, for instance to long-range initial conditions [69, 78], long-range interactions [67, 76, 78], external fields [70], conserved order parameters [62, 67, 71, 77], frustrations and/or disorder [64], external drives [81, 82] and so on. Several of those modifications could be of interest to experimentalists. It should be possible to generalise the Arcetri model in these directions, to work out the consequences on its ageing behaviour and explore relationships with long-range particle-reaction models [106].

Another important question will be if the Arcetri model can be understood as a $n \to \infty$ limit of a suitable n-component generalisation of the KPZ-equation, following the lines outlined in [107]. We hope to come back to this elsewhere.

Finally, the model analysed here is but one way to write down a 'spherical analogue' of the KPZ universality class. We used here the representation in terms of the Burgers equation

$$\partial_t u = \nu \partial_x^2 u + \mu u \partial_x u + \partial_x \eta \quad \mapsto \quad \partial_t u = \nu \partial_x^2 u + \mathfrak{z}(t) u + \partial_x \eta \tag{5.1}$$

where $\mathfrak{z}(t) \sim \langle \partial_x u \rangle \sim 1/t$ (which follows from $g(t) \sim t^F$) might be seen as some kind of 'averaged curvature' of the interface. However, the choice considered here was also motivated by its mathematical simplicity. It might be thought that a more faithful representation of the structure of the non-linear terms could be achieved in terms of a replacement

$$\partial_t u = \nu \partial_x^2 u + \mu u \partial_x u + \partial_x \eta \quad \mapsto \quad \partial_t u = \nu \partial_x^2 u + \mathfrak{z}(t) \partial_x u + \partial_x \eta \tag{5.2}$$

where $\mathfrak{z}(t) \sim \langle u \rangle$ might be viewed as some kind of 'averaged slope'. In these two cases, the Lagrange multiplier $\mathfrak{z}(t)$ is to be found from a spherical constraint $\sum_x \langle u^2 \rangle = \mathcal{N}$. Finally, we might have started directly from the KPZ equation

$$\partial_t h = \nu \partial_x^2 h + \frac{1}{2} \mu \left(\partial_x h \right)^2 + \eta \quad \mapsto \quad \partial_t h = \nu \partial_x^2 h + \mathfrak{z}(t) \partial_x h + \eta \tag{5.3}$$

where $\mathfrak{z}(t) \sim \langle \partial_x h \rangle$ might again be interpreted as an 'averaged slope' and will be found from a constraint $\sum_x \langle (\partial_x h)^2 \rangle = \mathcal{N}$. While eq. (5.1) has been studied here, work on the other two models, as defined in eqs. (5.2),(5.3) is in progress. The analysis of their sphericl constraints presents new features not seen in the system studied in this work and apparently leads to new types of behaviour. A sequel paper will take up the analysis of these distinct universality classes.

Appendix A. Computation of the long-time behaviour

The explicit long-time behaviour of the spherical constraint parameter g(t), for $T \leq T_c$ and the consequences for the interface width w(t) and the two-time responses and correlators, stated in section 4, will be derived.

Since $\overline{g}(p)$ is given by (4.1), we use standard Tauberian theorems [109, ch. XIII.5], in order to obtain the long-time behaviour of g(t) from the $p \to 0$ behaviour of $\overline{g}(p)$. To do so explicitly, two methods have been used: (i) one may consider g(t), f(t) as regular functions, which are found by inverse Laplace transformation. These must be completed by certain sum rules, needed for the computation of the correlators, see e.g. [64, 65, 67, 74]. (ii) perhaps more straightforwardly, accept that g(t), f(t) may have singular terms described by the Delta function and its derivatives, which are formally obtained by inverting the first few orders of $\overline{g}(p)$ [69, 79], see below. Although these singular terms do not contribute to the responses, they are important when inserted into the integrals which give the width or the correlators, since the singular terms in $g(\tau)$ will reproduce the effect of the sum rules just mentioned. In conclusion, both methods lead to the same final result.

A.1 Ageing at the critical point

Here, we consider the long-time behaviour at the critical point $T = T_c(d)$, as given by (4.4),(4.5).

In order to find $\overline{g}(p)$, we first expand $\overline{f}(p) = \int_0^\infty \mathrm{d}t \, e^{-pt} f(t)$, where f(t) was given in (3.12). In principle, such an expansion will contain terms analytic in p, i.e. $\sim p^n$ with $n \in \mathbb{N}$, and non-analytic contributions, i.e. $\sim p^\alpha$ with $\alpha \notin \mathbb{N}$. The origin of both is clearly seen by splitting the integral $\int_0^\infty \mathrm{d}t = \int_0^\eta \mathrm{d}t + \int_\eta^\infty \mathrm{d}t$ where η is a cut-off to be sent to infinity at the end. The first term can be formally expanded in p, as long as the corresponding coefficients converge, and gives the analytic part. The non-analytic part is found by inserting the $t \to \infty$ asymptotics of f(t) in the second term. To carry this out this, one sends first $p \to 0$ and only afterwards, one recovers the $\eta \to \infty$ limit. This standard calculation, e.g. [83, 108, 69], leads to the expansions, with $A_d^{-1} = -\frac{2}{\pi}(8\pi)^{d/2}\sin\left(\frac{\pi d}{2}\right)\Gamma\left(\frac{d}{2}\right)$, $\overline{f}(0) = d/(2T_c(d))$ and $\overline{f}'(0) = -\frac{d}{16}\int_0^\infty \mathrm{d}t \, e^{-dt}I_1(t)I_0(t)^{d-1}$

$$\overline{f}(p) \simeq \begin{cases}
\frac{d}{2T_c(d)} + A_d p^{d/2} & ; & \text{if } 0 < d < 2 \\
\frac{d}{2T_c(d)} + \frac{1}{16\pi} p \left(\ln p + C_E - 1 \right) & ; & \text{if } d = 2 \\
\frac{d}{2T_c(d)} + \overline{f}'(0) p + A_d p^{d/2} & ; & \text{if } 2 < d < 4
\end{cases}$$
(A.1)

 $(C_E = 0.5772...$ is Euler's constant) and for d > 4, a term $O(p^2)$ appears which does not contribute to the leading scaling behaviour. Hence, with $G_d := -\frac{H_1 d}{(2T_c(d))^2} \frac{1}{f'(0)} > 0$

$$\overline{g}(p) \simeq \begin{cases} -\frac{d}{(2T_c(d))^2} \frac{H_1}{A_d} p^{-d/2} - \frac{H_1}{2T_c(d)} + o(p) & ; & \text{if } 0 < d < 2\\ G_d p^{-1} - \frac{H_1}{2T_c(d)} + \frac{H_1 A_d}{(2T_c(d))^2} \frac{1}{(\overline{f}'(0))^2} p^{d/2 - 1} + O(p^{d - 2}) & ; & \text{if } 2 < d < 4 \end{cases}$$
(A.2)

and inverting this term-by-term gives, for $t\to\infty$ and with $g_d:=-\frac{d}{(2T_c(d))^2}\frac{H_1}{A_d}\frac{1}{\Gamma(d/2)}>0$

$$g(t) \simeq \begin{cases} g_d t^{d/2-1} - \frac{H_1}{2T_c(d)} \delta(t) & ; & \text{if } 0 < d < 2 \\ G_d - \frac{H_1}{2T_c(d)} \delta(t) + \mathcal{O}(t^{-d/2}) & ; & \text{if } d > 2 \end{cases}$$
(A.3)

In order to find the two-time response explicitly, it is enough to insert into (3.16) the asymptotic form of g(t) for $t \to \infty$, as derived above, to simply drop any distributional terms²³ and to expand $F_r(t)$ for large times as well.²⁴ This immediately gives (4.6),(4.7).

Next, we analyse the interface width w(t). Consider first the case 0 < d < 2. From (3.24), it follows

$$w^{2}(t) = \frac{H_{1}}{g(t)}F_{0}(2t) + \frac{2T_{c}}{g(t)}\int_{0}^{t} d\tau \ g(\tau)F_{0}(2t - 2\tau)$$

$$\simeq \frac{H_{1}}{g(t)}F_{0}(2t) + \frac{2T_{c}}{g(t)}\int_{0}^{t} d\tau \ \left[-\frac{H_{1}}{2T_{c}}\delta(\tau) - g_{d}\tau^{d/2-1} \right]F_{0}(2t - 2\tau)$$

$$\simeq 2T_{c}t^{1-d/2}\int_{0}^{t} d\tau \ \tau^{d/2-1}F_{0}(2t - 2\tau)$$

$$= 2T_{c}t^{1-d/2}\mathcal{L}^{-1}\left(\left(\mathcal{L}\tau^{d/2-1}\right)\left(p\right)\left(\mathcal{L}F_{0}(2\tau)\right)\left(p\right)\right)(t)$$

$$\simeq 2T_{c}\frac{\Gamma(1 - d/2)\Gamma(d/2)}{(8\pi)^{d/2}}\left(\mathcal{L}^{-1}\frac{1}{p}\right)(t)t^{1-d/2}$$

$$= \frac{2\pi}{(8\pi)^{d/2}\sin(\pi d/2)}T_{c}(d)t^{1-d/2}$$
(A.4)

which gives the first line in (4.11). In the second line in (A.4), we used in the integral the asymptotic form of $g(\tau)$, including both the singular distributional as well as the leading regular term. Integrating the singular term, we see that it cancels against the other contribution to $w^2(t)$. In the third line, we also inserted the asymptotic form of g(t). As for the response before, the singular terms do not contribute for $t \to \infty$. Furthermore, the non-universal amplitudes g_d contained in $g(\tau)$ and g(t) cancel. In the forth line, we recognise the remaining integral as a Laplace convolution which is treated via Laplace transforms. In the fifth line, we inserted the small-p behaviour of the two Laplace transforms, namely $(\mathcal{L}\tau^{d/2-1})$ $(p) = \Gamma(d/2)p^{-d/2}$ and

$$\left(\mathcal{L}F_{\mathbf{0}}(2t)\right)(p) = \int_{0}^{\infty} dt \ e^{-pt} \left(e^{-4t}I_{\mathbf{0}}(4t)\right)^{d} \stackrel{p\to 0}{\simeq} \int_{\eta}^{\infty} dt \ e^{-pt} (8\pi t)^{-d/2} + \mathrm{O}(1) \stackrel{p\to 0}{\simeq} \frac{\Gamma\left(1-d/2\right)}{(8\pi)^{d/2}} p^{d/2-1}$$

In analogy with the computation above of $\overline{f}(p)$, the integral diverges for d < 2 in the $p \to 0$ limit. Hence its leading contribution in this limit can be found by splitting the integral $\int_0^\infty = \int_0^\eta + \int_\eta^\infty$ with a cut-off η . The contribution of the first term remains finite for p > 0 and $\eta < \infty$ and it is enough here to extract the leading, divergent, behaviour of the second term. Herein, the Γ -functions are defined via analytic continuation, if necessary [89]. On the other hand, for d > 2, we first find the same cancellation of the H_1 -dependent term and then, using (A.3)

$$w^{2}(t) \simeq 2T_{c} \int_{0}^{t} d\tau \, F_{0}(2t - 2\tau) = 2T_{c} \underbrace{\int_{0}^{\infty} d\tau \, F_{0}(2\tau)}_{=:\mathcal{F}_{d}} - 2T_{c} \int_{t}^{\infty} d\tau \, F_{0}(2\tau) = 2T_{c}\mathcal{F}_{d} + O(t^{1-d/2})$$

since the amplitudes G_d cancel. The last integral was estimated by using $F_0(t) \sim t^{-d/2}$, for t large enough. This gives the last line in (4.11).

²³The singular terms in (A.3) do not contribute for t large. Heuristically, recall: $\delta(t) = \lim_{\lambda \to \infty} \sqrt{\lambda/\pi} e^{-\lambda t^2}$ vanishes for $t \neq 0$.

²⁴The relevant Bessel asymptotic formula is $I_r(t) \simeq (2\pi t)^{-1/2} e^{t-r^2/(2t)} (1 + \mathcal{O}(t^{-1}))$ for $t \to \infty$.

For the two-time correlator, consider first the case 0 < d < 2. We can use the same cancellation mechanism as before for the temperature-independent term and then find, assuming $t = ys \ge s$

$$C(t, s; \mathbf{r}) = \frac{H_1}{\sqrt{g(t)g(s)}} F_r(t+s) + \frac{2T_c}{\sqrt{g(t)g(s)}} \int_0^s d\tau \ g(\tau) F_r(t+s-2\tau)$$

$$\simeq \frac{H_1 F_r(t+s)}{\sqrt{g(t)g(s)}} - \frac{2T_c}{\sqrt{g(t)g(s)}} \frac{H_1}{2T_c} \int_0^s d\tau \ \delta(\tau) F_r(t+s-2\tau)$$

$$+2T_c(ts)^{(2-d)/4} \int_0^s d\tau \ \tau^{d/2-1} F_r(t+s-2\tau)$$

$$\stackrel{s\to\infty}{\simeq} s^{1-d/2} 2T_c y^{(2-d)/4} \int_0^1 du \ u^{d/2-1} \left(4\pi \left[(y+1) - 2u \right] \right)^{-d/2} \exp\left[-\frac{\mathbf{r}^2}{s} \frac{1}{y+1-2u} \right]$$
(A.6)

where in the second line the singular contribution in $g(\tau)$ from (A.3) has been explicitly introduced to demonstrate the cancellation of the H_1 -dependent terms. In the third line, the regular large-time asymptotics of g(t) has been inserted and finally, $F_r(t)$ is expanded for large times. For d > 2, we now have $g(t) \sim t^0$ instead of $t^{d/2-1}$ and an analogous calculation gives

$$C(t,s; \mathbf{r}) = s^{1-d/2} 2T_c \int_0^1 du \, u^{d/2-1} \left(4\pi \left[(y+1) - 2u \right] \right)^{-d/2} \exp \left[-\frac{\mathbf{r}^2}{s} \frac{1}{y+1-2u} \right]$$
(A.7)

which proves the assertion in section 4. For r = 0, the two-time autocorrelator and its scaling function (4.10) is readily obtained.

Finally, we examine the special case d=2. Indeed, the treatment is analogous to the one of the 4D magnetic spherical model [74, 79]. From (A.1), one readily finds $\overline{g}(p)$. For large times, it can be shown that $g(t) \simeq -H_1/(2T_c)\delta(t) + \overline{g}_2/\ln t$ [74, 79, 109]. The spherical parameter g(t) enters into the two-time response and correlation functions only through ratios $g(t)/g(s) \simeq \ln t/\ln s = \ln(ys)/\ln s \simeq 1 + \ln y/\ln s$. Therefore, these logarithmic terms only arise as additive logarithmic corrections to scaling, but do not affect the scaling form itself [74, 79]. This can also be seen in the explicit results (4.7),(4.10), since the scaling functions are continuous in d, even at d=2. This is different, however, for the interface width, where the amplitudes eq. (4.11) diverge for $d \to 2$. This case must therefore be analysed separately. In analogy with the treatment which led to (A.4) for d < 2, we have first the cancellation of the H_1 -dependent terms and then estimate the large-t behaviour of the remaining integral

$$w^{2}(t) \simeq 2T_{c} \ln t \int_{0}^{t} d\tau \frac{1}{\ln \tau} F_{\mathbf{0}}(2t - 2\tau) = 2T_{c} \ln t \mathcal{L}^{-1} \left(\left(\mathcal{L} \left(\frac{1}{\ln \tau} \right) \right) (p) \left(\mathcal{L} F_{\mathbf{0}}(2\tau) \right) (p) \right) (t)$$

$$\simeq \frac{2T_{c}}{8\pi} \ln t \left(\mathcal{L}^{-1} \frac{1}{p} \right) (t) \sim \ln t$$
(A.8)

Again, the non-universal amplitude \bar{g}_2 cancels. We recognise the integral as a Laplace convolution and estimate the leading small-p behaviour of the two factors. First, we recall $(\mathcal{L}(1/\ln \tau))(p) \sim -[p(C_E + \ln p)]^{-1}$, see [74, 79]. The second factor is the Laplace transform of $F_0(2\tau)$ in 2D, which is evaluated in analogy with similar integrals treated above, as follows (with a cut-off η ; E_1 is the exponential integral [89, eq. (5.1.11)])

$$(\mathcal{L}F_{\mathbf{0}}(2\tau))(p) = \int_{0}^{\infty} d\tau \ e^{-(p+8)\tau} I_{0}^{2}(4\tau)) \overset{p\to 0}{\simeq} \int_{\eta}^{\infty} d\tau \ e^{-p\tau} (8\pi\tau)^{-1} = \frac{1}{8\pi} E_{1}(p\eta) \overset{p\to 0}{\simeq} -\frac{C_{E} + \ln p}{8\pi}$$

As in [79], terms only depending on $\ln \eta$ must be absorbed into the 'finite' contribution coming from the integral \int_0^{η} , which leads here to non-leading terms. This proves the middle line in (4.11).

A.2 Ageing at low temperatures

For $T < T_c$, define the auxiliary variable $m^2 := 1 - T/T_c$. First, we find $\overline{g}(p)$ from (4.1), with $\overline{f}(p)$ explicitly given in (A.1). This gives

$$\overline{g}(p) \simeq \begin{cases} \frac{H_1}{2T_c} \frac{1}{m^2} + \frac{H_1 A_d}{dm^4} p^{d/2} + \dots & ; & \text{if } 0 < d < 2\\ \frac{H_1}{2T_c} \frac{1}{m^2} + \frac{H_1}{m^4} \overline{f}'(0) p + \frac{H_1 A_d}{dm^4} p^{d/2} + \dots & ; & \text{if } 2 < d < 4 \end{cases}$$
(A.9)

where we used $T_c + T/m^2 = T_c/m^2$. Further regular terms appear for d > 4, 6, ..., but will only give rise to corrections to the leading scaling behaviour. This gives in turn

$$g(t) \simeq \begin{cases} \frac{H_1}{2T_c} \frac{1}{m^2} \delta(t) + \frac{H_1}{4m^4} (8\pi)^{-d/2} t^{-1-d/2} + \dots & ; & \text{if } 0 < d < 2\\ \frac{H_1}{2T_c} \frac{1}{m^2} \delta(t) + \frac{H_1}{dm^4} \overline{f}'(0) \delta'(t) + \frac{H_1}{4m^4} (8\pi)^{-d/2} t^{-1-d/2} + \dots & ; & \text{if } 2 < d < 4 \end{cases}$$
(A.10)

The response function is once more obtained by straightforward asymptotic expansion, which leads again to (4.6) with the scaling function $f_R(y)$ being now given by (4.18).

Next, we analyse the interface width. The techniques to be used are quite close to the one applied in the critical case, but the leading term turns out to be of a different form (let d < 2 for simplicity)

$$w^{2}(t) = \frac{H_{1}F_{0}(2t)}{g(t)} + 2T \int_{0}^{t} d\tau \, \frac{g(\tau)}{g(t)} F_{0}(2t - 2\tau)$$

$$\simeq \frac{H_{1}}{g(t)} \left(1 + \frac{1}{m^{2}} \frac{T}{T_{c}} \right) F_{0}(2t) + 2T \int_{0}^{t} d\tau \, \left(\frac{\tau}{t} \right)^{-1 - d/2} F_{0}(2t - 2\tau)$$

$$\simeq 4m^{2}t + 2Tt^{1 + d/2} \mathcal{L}^{-1} \left(\left(\mathcal{L}(\tau^{-1 - d/2})(p) \right) \left(\mathcal{L}F_{0}(2\tau)(p) \right) \right) (t)$$

$$= 4m^{2}t + 2Tt^{1 + d/2} \frac{\Gamma(-d/2)\Gamma(1 - d/2)}{(8\pi)^{d/2}} \left(\mathcal{L}^{-1}p^{d-1} \right) (t)$$

$$= 4m^{2}t + O\left(Tt^{1 - d/2}\right)$$
(A.11)

In the second line, we used both the singular and the leading regular term for $g(\tau)$. In contrast to the critical case, the singular contribution does not cancel with the other H_1 -dependent term, but combines into a new, leading term which depend on the ratio T/T_c and vanishes when $T \to T_c$. Therefore, the leading long-time behaviour only depends on the ratio T/T_c and the short-ranged nature of the initial correlations. The 'thermal' convolution term can be calculated via a Laplace convolution as before, but turns out to provide merely a correction to scaling. For d > 2, the leading correction is O(T) and the further derivatives of the δ -function only generate, via partial integrations, further sub-leading corrections to scaling. Hence the leading result $w^2(t) \simeq 4m^2t$, valid for t large enough, is independent of d, as asserted in (4.21).

The two-time autocorrelator is analysed in the same way (assume d < 2 and let y = t/s > 1)

$$C(t,s) = \frac{H_1 F_0(t+s)}{\sqrt{g(t)g(s)}} + 2T \int_0^s d\tau \, \frac{g(\tau)}{\sqrt{g(t)g(s)}} F_0(t+s-2\tau)$$

$$\simeq \frac{m^4 \Gamma(-d/2) d}{A_d (4\pi)^{d/2}} \frac{(ts)^{(2+d)/4}}{(t+s)^{d/2}} \left(1 + \frac{1}{m^2} \frac{T}{T_c}\right) + \frac{2T}{(4\pi)^{d/2}} (ts)^{(2+d)/4} \int_0^s d\tau \, \tau^{-1-d/2} (t+s-2\tau)^{-d/2}$$

$$= 2^{d+2} m^2 \, s \, y^{(2+d)/4} (y+1)^{-d/2} + \mathcal{O}\left(Ts^{1-d/2}\right) \tag{A.12}$$

as stated in (4.19). For d > 2, the leading correction is O(T).

Since all results are continuous in d at d=2, a separate analysis of the 2D case is not necessary.

Appendix B. On the Yeung-Rao-Desai inequalities

The analogue of the well-known YRD-inequality [104], originally formulated for ageing magnetic systems, is: for a growing interface with a non-conserved dynamics and an uncorrelated initial state, the autocorrelation exponent λ_C satisfies the bound

$$\lambda_C \ge \frac{1}{2} \left(d + zb \right) \tag{B.1}$$

Such bounds may serve as checks on exponents, estimated from simulational or experimental data (i.e. for phase-ordering magnets with $T < T_c$, one has b = 0, hence $\lambda_C \ge d/2$ [104]).

To see this, recall first the argument [104] to bound the two-time height autocorrelator

$$C(t,s) = \int d\mathbf{k} \, \widehat{C}(t,s;\mathbf{k},-\mathbf{k}) \le \int d\mathbf{k} \, \sqrt{\widehat{C}(t,\mathbf{k}) \, \widehat{C}(s,\mathbf{k})}$$
(B.2)

Eq. (1.2) gives the scaling form of the single-time correlator $\widehat{C}(t, \mathbf{k}) = \widehat{C}(t, t; \mathbf{k}, -\mathbf{k})$

$$\widehat{C}(t, \boldsymbol{k}) = \frac{1}{\sqrt{V}} \int_{V} d\boldsymbol{r} \, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} \, t^{-b} F_{C}\left(1; \boldsymbol{r} t^{-1/z}\right) = L(t)^{d-zb} \, \mathcal{C}\left(kL(t)\right)$$

where the typical length scale $L(t) \sim t^{1/z}$ and \mathcal{C} is a scaling function. In order to estimate the integral, recall that the height fluctuations $\delta \widehat{h}(t, \mathbf{k})$ and $\delta \widehat{h}(s, -\mathbf{k})$ in $\widehat{C}(t, s; \mathbf{k}, -\mathbf{k})$ should become uncorrelated over distances larger than $\Delta L \gtrsim 2\pi \mathbf{a}/|\mathbf{k}|$ such that $\widehat{C}(t, s; \mathbf{k}, -\mathbf{k}) \to 0$ rapidly if $(L(t) - L(s)) |\mathbf{k}| \gg 1$ [104]. For uncorrelated initial conditions, one has $\lim_{s\to 0} \widehat{C}(s, \mathbf{k}) \sim \lim_{\mathbf{k}\to 0} \widehat{C}(s, \mathbf{k}) = \mathrm{O}(1)$. In the scaling limit with y = t/s large enough, the bound (B.2) gives

$$\begin{split} &\lim_{t,s\to\infty} C(t,s) \sim L(t)^{-\lambda_C} \\ &\leq \ L(t)^{(d-zb)/2} \, C_d \int_0^{2\pi \mathsf{a}/L(t)} \! \mathrm{d} k \, k^{d-1} \, \mathcal{C}^{1/2} \left(kL(t) \right) \, = \, L(t)^{-(d+zb)/2} \, C_d \int_0^{2\pi \mathsf{a}} \! \mathrm{d} u \, u^{d-1} \, \mathcal{C}^{1/2}(u) \end{split}$$

 $(C_d \text{ is a constant}) \text{ and (B.1) follows.}$

Alternatively, if the waiting time s is itself already in the scaling regime, one has $\widehat{C}(s, \mathbf{k}) \stackrel{|\mathbf{k}| \to 0}{\sim} k^{zb-d}$ which leads to $\lambda_C \geq zb$, with an magnetic analogue already given in [104].

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